

Anisotropic hydrodynamics: A progress report

Michael Strickland
Kent State University
Kent, OH USA

Opportunities for Exploring Longitudinal Dynamics in Heavy Ion Collisions at RHIC
RIKEN BNL Research Center (RBRC)
January 21, 2016

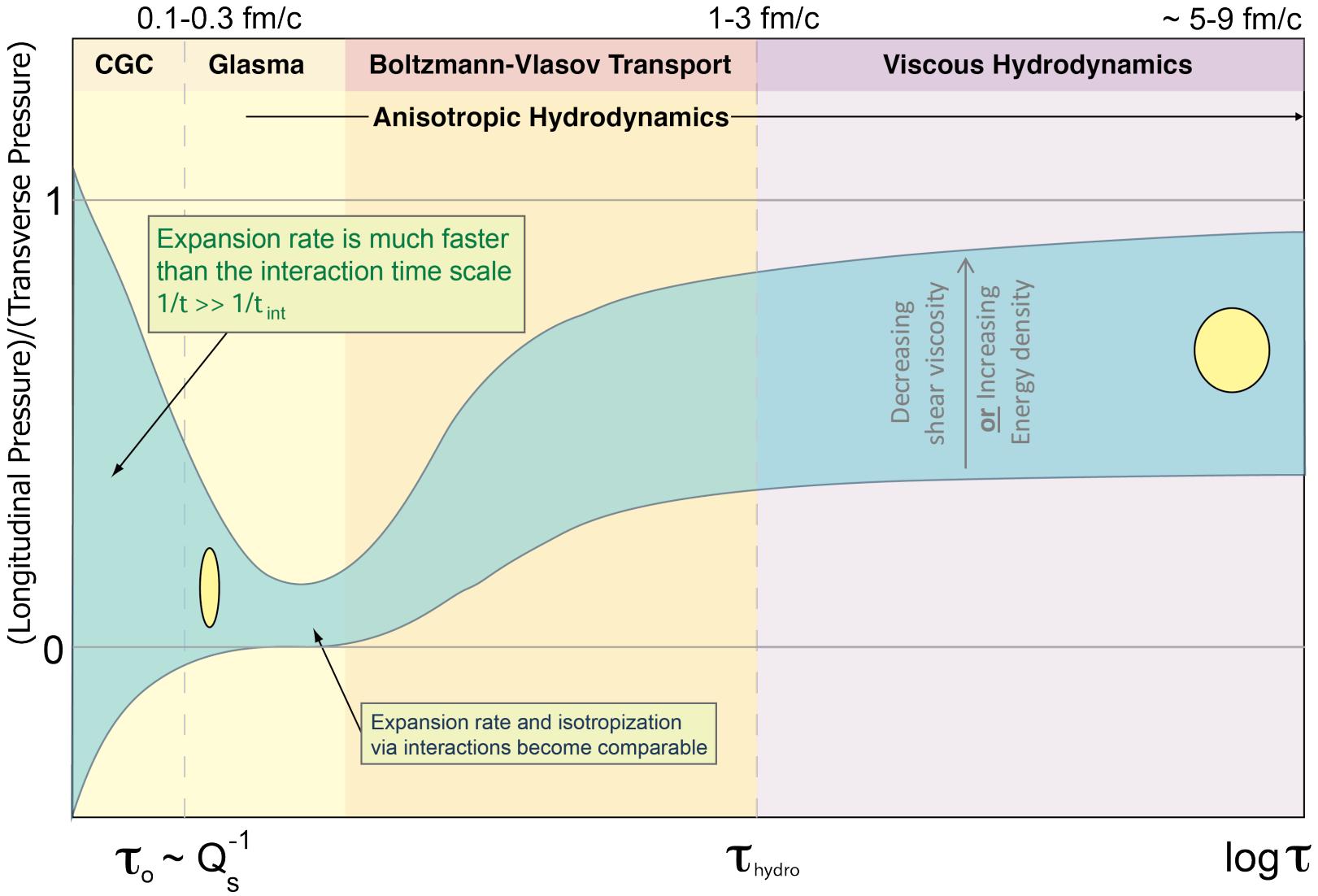
Recent Review: M. Strickland, 1410.5786



Motivation

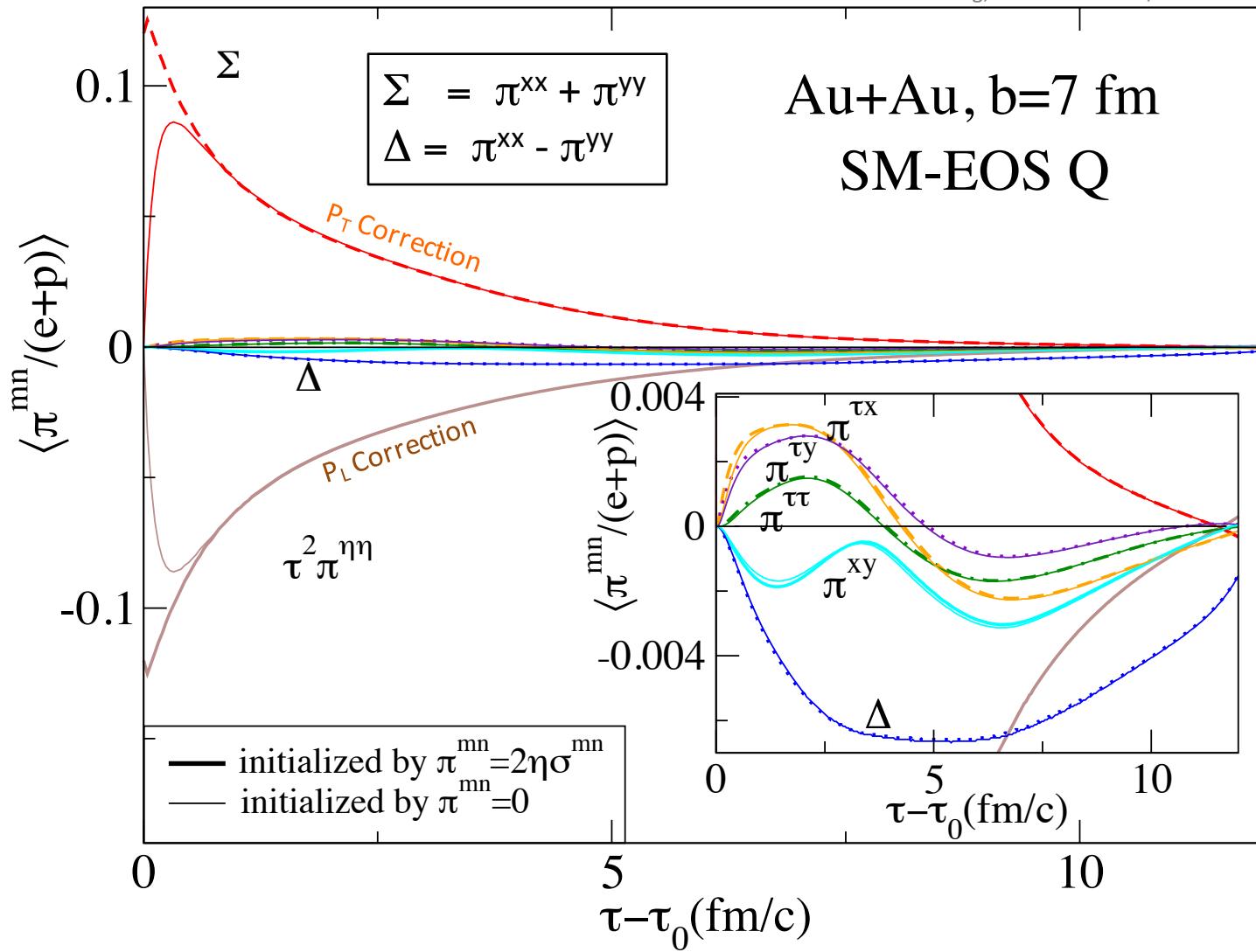
- Application of relativistic viscous hydro justified a priori by the smallness of the shear viscosity of the plasma (relative to the entropy density)
- The canonical way to derive viscous hydrodynamics relies on a linearization around an isotropic equilibrium state (local rest frame = LRF)
- However, the QGP is not isotropic in LRF → there are large corrections to ideal hydrodynamics due to strong longitudinal expansion
- **Alternative approach:** Anisotropic hydrodynamics builds in momentum-space anisotropies in the LRF from the beginning
- The goal is to create a quantitatively reliable viscous-hydro-like code that more accurately describes:
 - Early time dynamics
 - Small systems
 - Dynamics near the transverse edges of the overlap region (dilute)
 - Dynamics at forward rapidity (dilute)
 - Temperature-dependent (and potentially large) η/S

QGP momentum anisotropy cartoon



Hints from Viscous Hydro

H. Song, PhD Dissertation, 0908.3656



Anisotropic hydrodynamics beginning

[M. Martinez and MS, 1007.0889]

[W. Florkowski and R. Ryblewski, 1007.0130]

Viscous Hydrodynamics Expansion

$$f(\tau, \mathbf{x}, \mathbf{p}) = \underline{f_{\text{eq}}(\mathbf{p}, T(\tau, \mathbf{x}))} + \delta f$$

↑
Isotropic in momentum space

Anisotropic Hydrodynamics (aHydro) Expansion

$$f(\tau, \mathbf{x}, \mathbf{p}) = f_{\text{aniso}}(\mathbf{p}, \underbrace{\Lambda(\tau, \mathbf{x})}_{T_\perp}, \underbrace{\xi(\tau, \mathbf{x})}_{\text{anisotropy}}) + \delta \tilde{f}$$

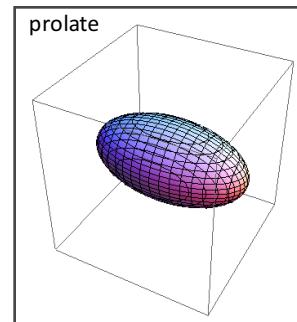
Treat this term perturbatively
→ “NLO aHydro”

D. Bazow, U. Heinz, and MS, 1311.6720
D. Bazow, U. Heinz, and M. Martinez, 1503.07443

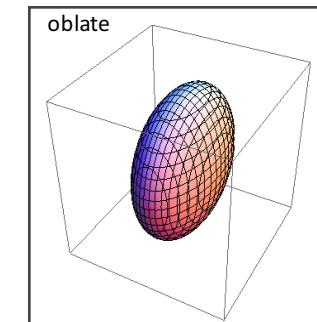
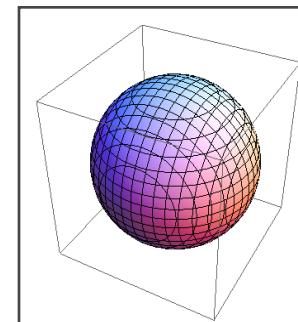
→ “Romatschke-Strickland” form in LRF

$$f_{\text{aniso}}^{\text{LRF}} = f_{\text{iso}} \left(\frac{\sqrt{\mathbf{p}^2 + \xi(\mathbf{x}, \tau)p_z^2}}{\Lambda(\mathbf{x}, \tau)} \right)$$

$$\xi = \frac{\langle p_T^2 \rangle}{2 \langle p_L^2 \rangle} - 1$$



$$-1 < \xi < 0$$



$$\xi > 0$$

Why spheroidal form at LO?

- What is special about this form at leading order?

$$f_{\text{aniso}}^{LRF} = f_{\text{iso}} \left(\frac{\sqrt{\mathbf{p}^2 + \xi(\mathbf{x}, \tau) p_z^2}}{\Lambda(\mathbf{x}, \tau)} \right)$$

- Gives the ideal hydro limit when $\xi=0$ ($\Lambda \rightarrow T$)
- For longitudinal (0+1d) free streaming, the LRF distribution function is of spheroidal form; limit emerges automatically in aHydro

$$\xi_{\text{FS}}(\tau) = (1 + \xi_0) \left(\frac{\tau}{\tau_0} \right)^2 - 1$$

- Since $f_{\text{iso}} \geq 0$, the one-particle distribution function and pressures are ≥ 0 (not guaranteed in viscous hydro)
- 0+1d equations reduce to 2nd-order viscous hydrodynamics in limit of small anisotropies

$$\frac{\Pi}{\mathcal{E}_{\text{eq}}} = \frac{8}{45} \xi + \mathcal{O}(\xi^2)$$

For 3+1d proof of equivalence to second-order viscous hydrodynamics in the near-equilibrium limit see Tinti 1411.7268.

NLO Anisotropic Hydrodynamics

[D. Bazow, U. Heinz, and MS, 1311.6720]

[D. Bazow, U. Heinz, and M. Martinez, 1503.07443]

$$f(\tau, \mathbf{x}, \mathbf{p}) = f_{\text{aniso}}(\mathbf{p}, \underbrace{\Lambda(\tau, \mathbf{x})}_{T_\perp}, \underbrace{\xi(\tau, \mathbf{x})}_{\text{anisotropy}}) + \tilde{\delta f}$$

- Treat LO term “non-perturbatively” (all orders in ξ) assuming spheroidal form and couple it to the dissipative currents
- Treat corrections $\tilde{\delta f}$ “perturbatively” \rightarrow NLO aHydro
- Use the method of Denicol et al [1202.4551] adapted to an anisotropic background
- Complete and orthogonal relativistic polynomial basis + systematic expansion in Knudsen number and (modified) inverse Reynolds number

The resulting equations of motion

[D. Bazow, U. Heinz, and MS, 1311.6720]

First moment

$$\begin{aligned}\dot{\mathcal{E}} + (\mathcal{E} + \mathcal{P}_\perp + \tilde{\Pi})\theta + (\mathcal{P}_L - \mathcal{P}_\perp)\frac{u_0}{\tau} + u_\nu \partial_\mu \tilde{\pi}^{\mu\nu} &= 0, \\ (\mathcal{E} + \mathcal{P}_\perp + \tilde{\Pi})\dot{u}_x + \partial_x(\mathcal{P}_\perp + \tilde{\Pi}) + u_x(\dot{\mathcal{P}}_\perp + \dot{\tilde{\Pi}}) + (\mathcal{P}_\perp - \mathcal{P}_L)\frac{u_0 u_x}{\tau} - \Delta^{1\nu} \partial^\mu \tilde{\pi}_{\mu\nu} &= 0, \\ (\mathcal{E} + \mathcal{P}_\perp + \tilde{\Pi})\dot{u}_y + \partial_y(\mathcal{P}_\perp + \tilde{\Pi}) + u_y(\dot{\mathcal{P}}_\perp + \dot{\tilde{\Pi}}) + (\mathcal{P}_\perp - \mathcal{P}_L)\frac{u_0 u_y}{\tau} - \Delta^{2\nu} \partial^\mu \tilde{\pi}_{\mu\nu} &= 0,\end{aligned}$$

Second moment
(anisotropic Grad-14
approximation)

$$\begin{aligned}\dot{\tilde{V}}^{\langle\mu\rangle} &= \mathcal{C}_{-1}^{\langle\mu\rangle} + \mathcal{Z}^\mu - \tilde{V}^\lambda \nabla_\lambda u^\mu - \tilde{V}^\mu \theta - \ell_{V\Pi}^{\mu\nu} \nabla_\nu \tilde{\Pi} - \tau_{V\Pi}^\mu \tilde{\Pi} - \delta_{VV}^{\mu\nu\alpha\beta} \tilde{V}_\nu \nabla_\alpha u_\beta \\ &\quad + \ell_{V\pi}^{\mu\mu\alpha\beta} \nabla_\nu \tilde{\pi}_{\alpha\beta} + \tau_{V\pi}^{\mu\alpha\beta} \tilde{\pi}_{\alpha\beta}, \\ \dot{\tilde{\pi}}^{\langle\mu\nu\rangle} &= \mathcal{C}_{-1}^{\langle\mu\nu\rangle} + \mathcal{K}^{\mu\nu} + \mathcal{L}^{\mu\nu} + \mathcal{H}^{\mu\nu\lambda} (\dot{z}_\lambda + u^\alpha \nabla_\lambda z_\alpha) + \mathcal{Q}^{\mu\nu\lambda\alpha} \nabla_\lambda u_\alpha \\ &\quad - \frac{5}{3} \tilde{\pi}^{\mu\nu} \theta - 2\tilde{\pi}_\lambda^{\langle\mu} \sigma^{\nu\rangle\lambda} + 2\tilde{\pi}_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} + 2\tilde{\Pi} \sigma^{\mu\nu} \\ &\quad - \ell_{\pi V}^{\mu\nu\alpha\beta} \nabla_\alpha \tilde{V}_\beta - \tau_{\pi V}^{\mu\nu\lambda} \tilde{V}_\lambda - \tilde{\Pi} \delta_{\pi\Pi}^{\mu\nu\alpha\beta} \nabla_\alpha u_\beta - \delta_{\pi\pi}^{\mu\nu\alpha\beta\sigma\lambda} \tilde{\pi}_{\sigma\lambda} \nabla_\alpha u_\beta. \\ -\frac{3}{m^2} \dot{\tilde{\Pi}} &= \mathcal{C}_{-1} + \mathcal{W} + \beta_{\Pi\perp} \theta + \beta_{\Pi L} z^\mu z^\nu \sigma_{\mu\nu} - \tilde{\Pi} \theta - \lambda_{\Pi V}^{\mu\nu} \nabla_\mu \tilde{V}_\nu - \tau_{\Pi V}^\mu \tilde{V}_\mu \\ &\quad - \delta_{\Pi\Pi}^{\mu\nu} \tilde{\Pi} \nabla_\mu u_\nu - \tilde{\pi}_{\alpha\beta} \delta_{\Pi\pi}^{\mu\nu\alpha\beta} \nabla_\mu u_\nu.\end{aligned}$$

For the extension to non-conformal systems see Bazow, Heinz, and Martinez, 1503.07443.

Beauty vs the Beast

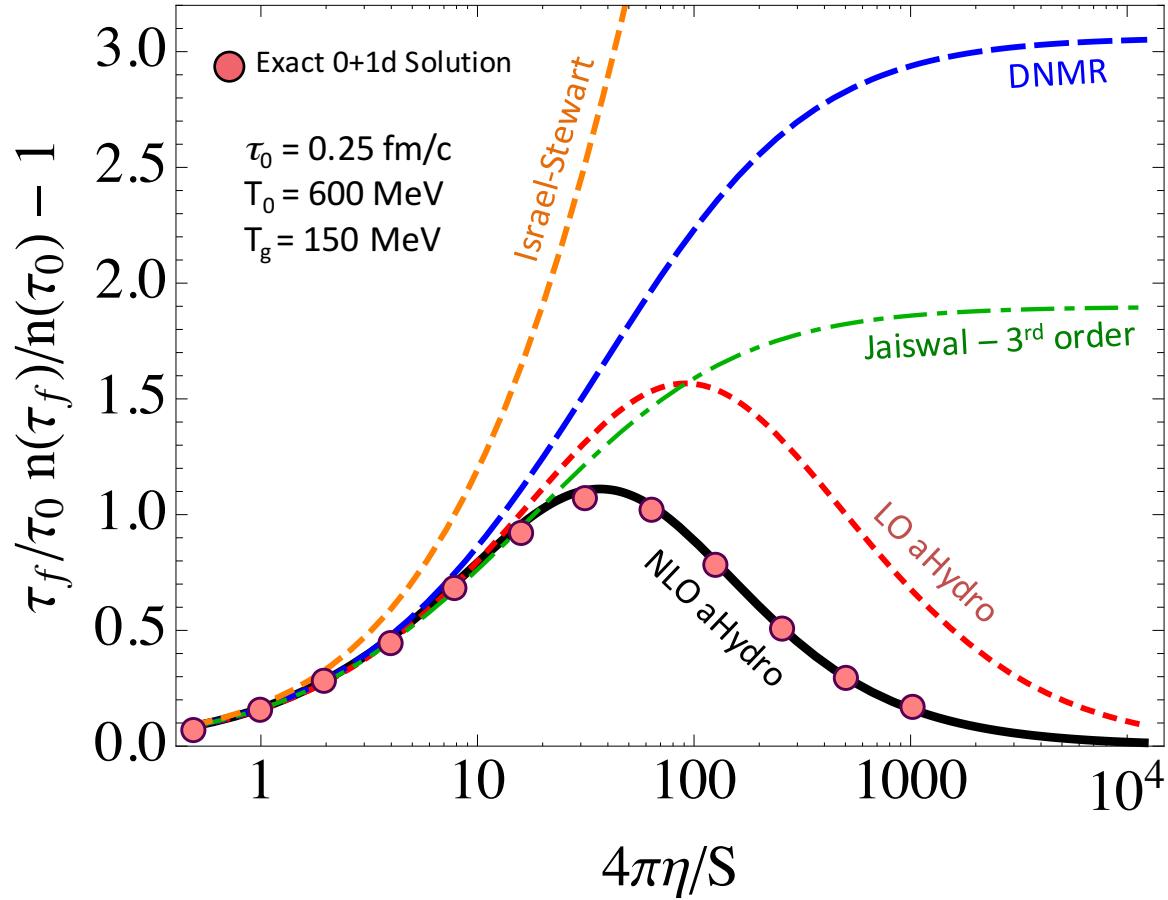
- You might say at this point, “But Mike, aren’t complicated things inherently bad?”
- Perhaps, but sometimes they are necessary.
- Comparison to exact solutions of the RTA Boltzmann equation show that aHydro seems to work better than conventional dissipative hydro approaches.



Conformal 0+1d results

[W. Florkowski, R. Ryblewski, and MS, 1304.0665 and 1305.7234]

[D. Bazow, U. Heinz, and MS, 1311.6720]

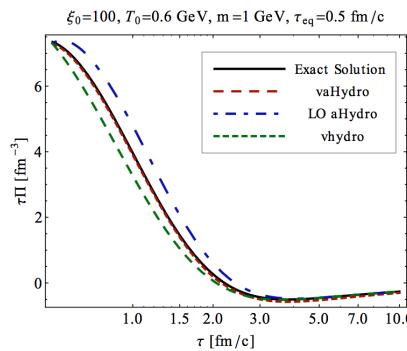
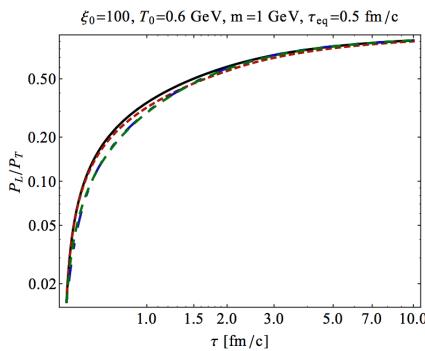
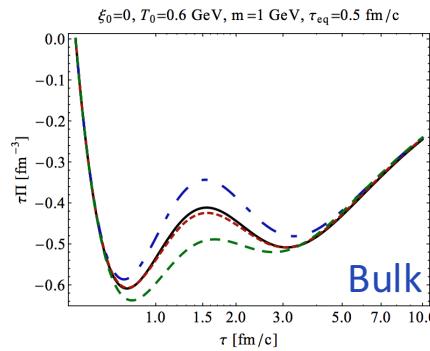
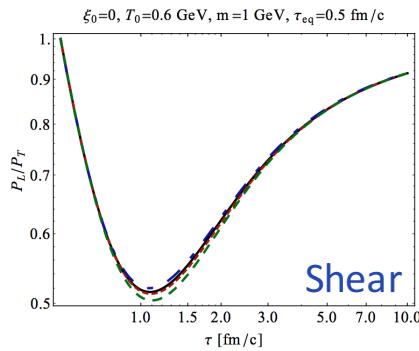


- Number (entropy) production vanishes in two limits: ideal hydrodynamic and free streaming limits
- In the conformal model which we are testing with, number density is proportional to entropy density

0+1d non-conformal (massive) gas

[W. Florkowski, E. Maksymiuk, R. Ryblewski, and MS, 1402.7348]

$$2m^2T(\tau) \left[3T(\tau)K_2\left(\frac{m}{T(\tau)}\right) + mK_1\left(\frac{m}{T(\tau)}\right) \right] \\ = D(\tau, \tau_0)\Lambda_0^4 \tilde{\mathcal{H}}_2 \left[\frac{\tau_0}{\tau\sqrt{1+\xi_0}}, \frac{m}{\Lambda_0} \right] + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\text{eq}}} D(\tau, \tau') T^4(\tau') \tilde{\mathcal{H}}_2 \left[\frac{\tau'}{\tau}, \frac{m}{T(\tau')} \right]$$



- Can also obtain the exact kinetic solution for a massive gas
- Allows one to assess different methods for inclusion of bulk viscous effects
- The overarching conclusion is that it is important to include shear-bulk couplings (Israel-Stewart fails completely).

M. Nopoush, R. Ryblewski, and MS, 1405.1355

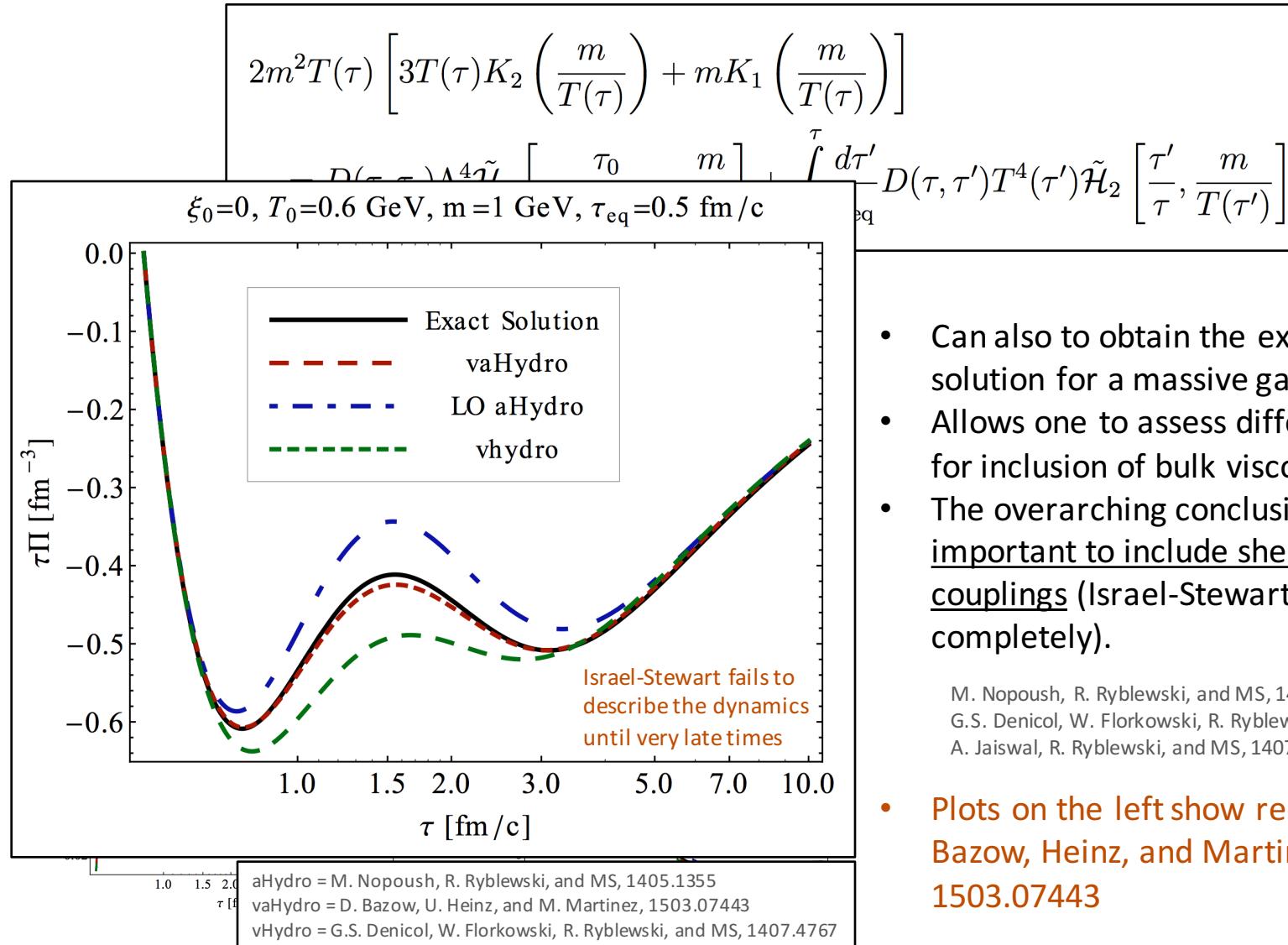
G.S. Denicol, W. Florkowski, R. Ryblewski, and MS, 1407.4767

A. Jaiswal, R. Ryblewski, and MS, 1407.7231

- Plots on the left show recent results of Bazow, Heinz, and Martinez
1503.07443

0+1d non-conformal (massive) gas

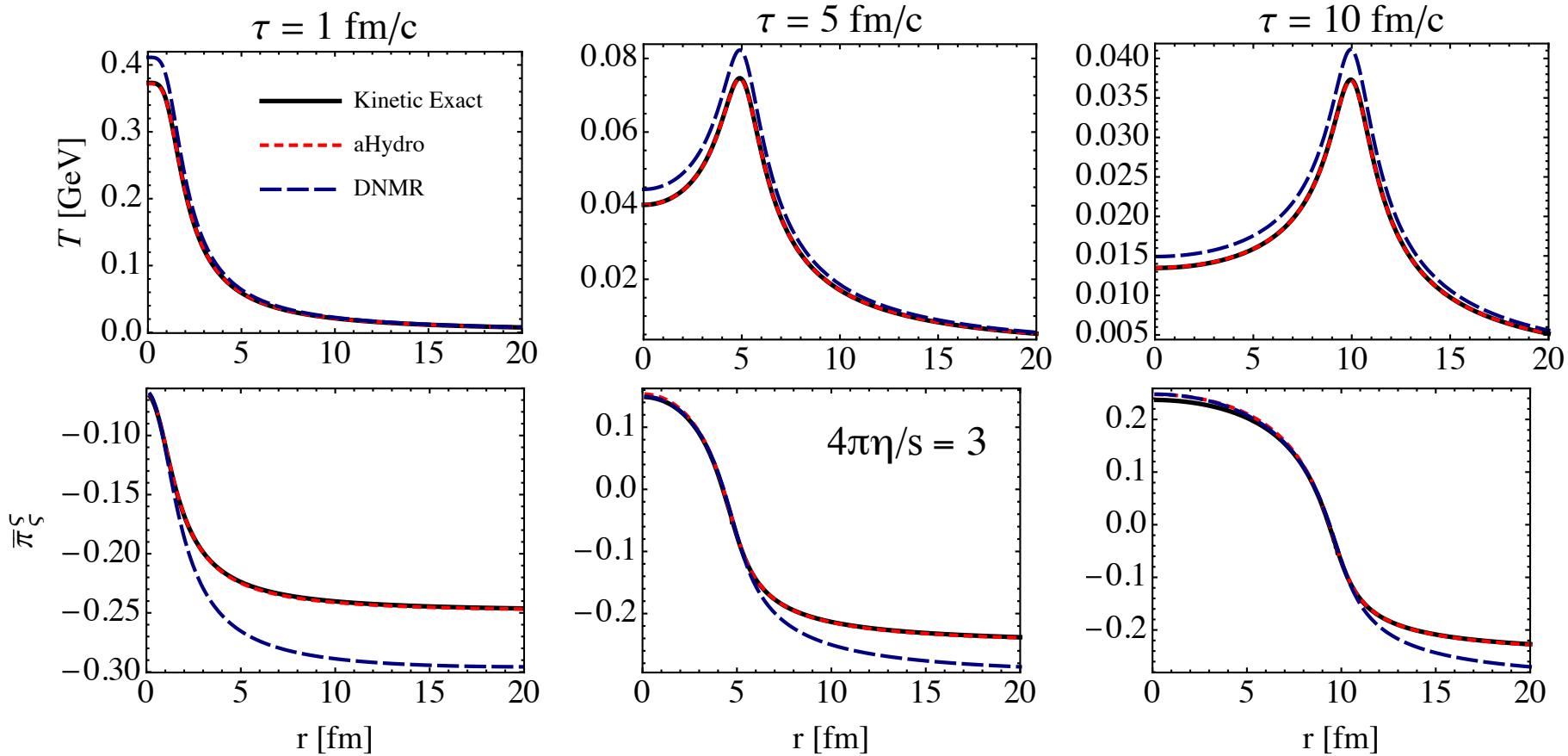
[W. Florkowski, E. Maksymiuk, R. Ryblewski, and MS, 1402.7348]



1+1d exact solution for Gubser Flow

M. Nopoush, R. Ryblewski, and MS, 1410.6790

Exact kinetic solution: G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048



Making aHydro ready for primetime

- Generalized LO spheroidal form to ellipsoidal form
- Implement realistic lattice-based equation of state; not trivial to think about in an anisotropic system
- Implement self-consistent anisotropic Cooper-Frye freeze-out including conformal breaking (bulk effects)

A more general version of aHydro

L. Tinti and W. Florkowski, 1312.6614 (massless)
M. Nopoush, R. Ryblewski, and MS, 1405.1355 (massive)

In generalized aHydro, one assumes that the distribution function is of the form

$$f(x, p) = f_{\text{eq}} \left(\frac{\sqrt{p^\mu \Xi_{\mu\nu}(x) p^\nu}}{\lambda(x)}, \frac{\mu(x)}{\lambda(x)} \right) + \delta \tilde{f}(x, p)$$

$$\Xi^{\mu\nu} = \underbrace{u^\mu u^\nu}_{\text{LRF four velocity}} + \underbrace{\xi^{\mu\nu}}_{\substack{\text{traceless} \\ \text{anisotropy} \\ \text{tensor}}} - \underbrace{\Delta^{\mu\nu} \Phi}_{\substack{\text{“Bulk”} \\ \uparrow \\ \text{Transverse projector}}}$$

$$\begin{aligned} u^\mu u_\mu &= 1 \\ \xi^\mu{}_\mu &= 0 \\ \Delta^\mu{}_\mu &= 3 \\ u_\mu \xi^{\mu\nu} &= u_\mu \Delta^{\mu\nu} = 0 \end{aligned}$$

Implementing the equation of state

[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, 1506.05278]
[Alqahtani, Nopoush, MS, 1509.02913]

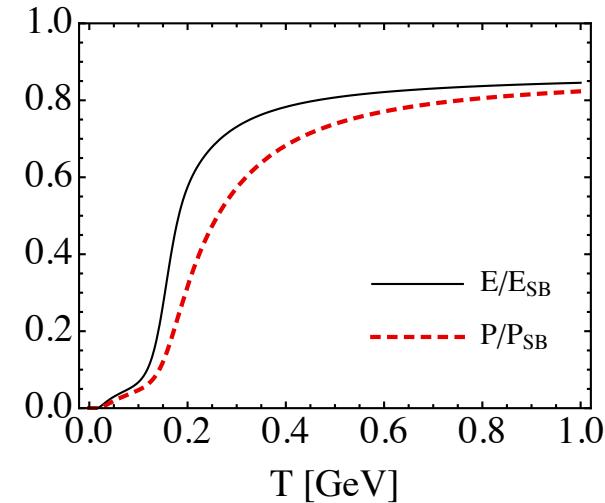
Standard Method

$$n(\Lambda, \xi) = \int \frac{d^3 p}{(2\pi)^3} f_{\text{aniso}} = \frac{n_{\text{iso}}(\Lambda)}{\sqrt{1 + \xi}}$$

$$\mathcal{E}(\Lambda, \xi) = T^{\tau\tau} = \mathcal{R}(\xi) \mathcal{E}_{\text{iso}}(\Lambda)$$

$$\mathcal{P}_\perp(\Lambda, \xi) = \frac{1}{2} (T^{xx} + T^{yy}) = \mathcal{R}_\perp(\xi) \mathcal{P}_{\text{iso}}(\Lambda)$$

$$\mathcal{P}_L(\Lambda, \xi) = -T_\zeta^\zeta = \mathcal{R}_L(\xi) \mathcal{P}_{\text{iso}}(\Lambda)$$

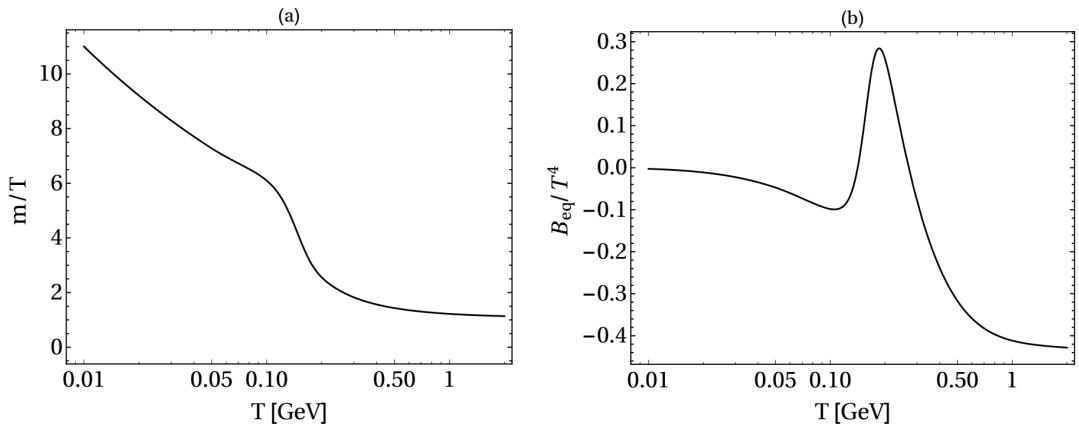


Quasiparticle Method

$$T_{\text{eq}}^{\mu\nu} = T_{\text{kinetic, eq}}^{\mu\nu} + g^{\mu\nu} B_{\text{eq}}$$

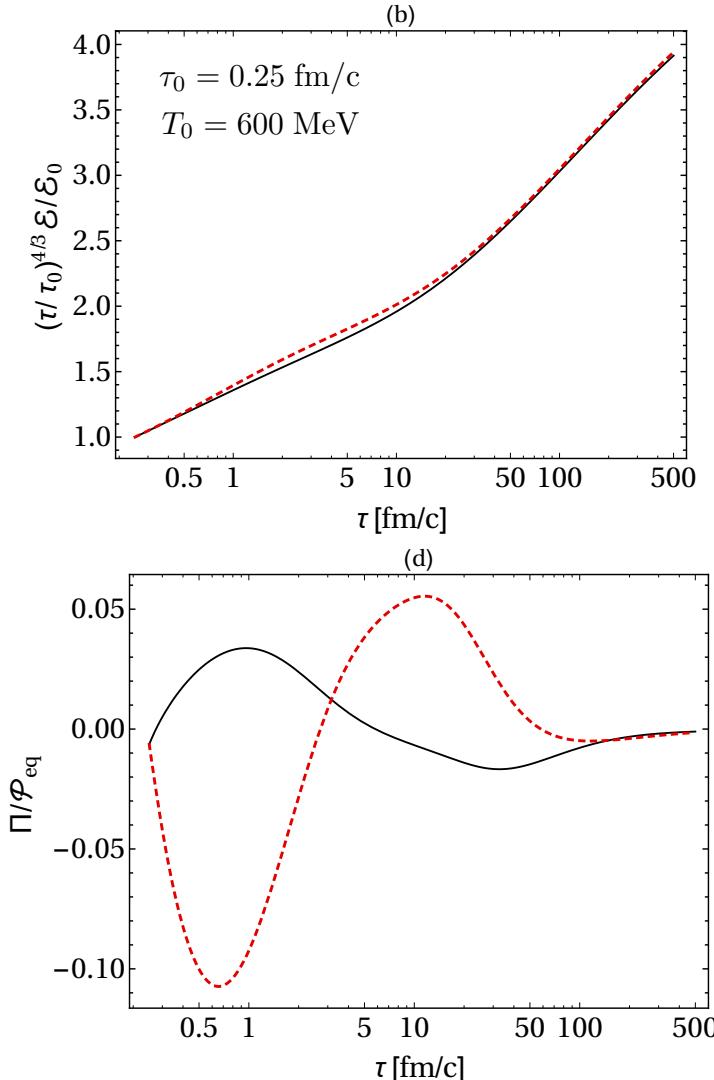
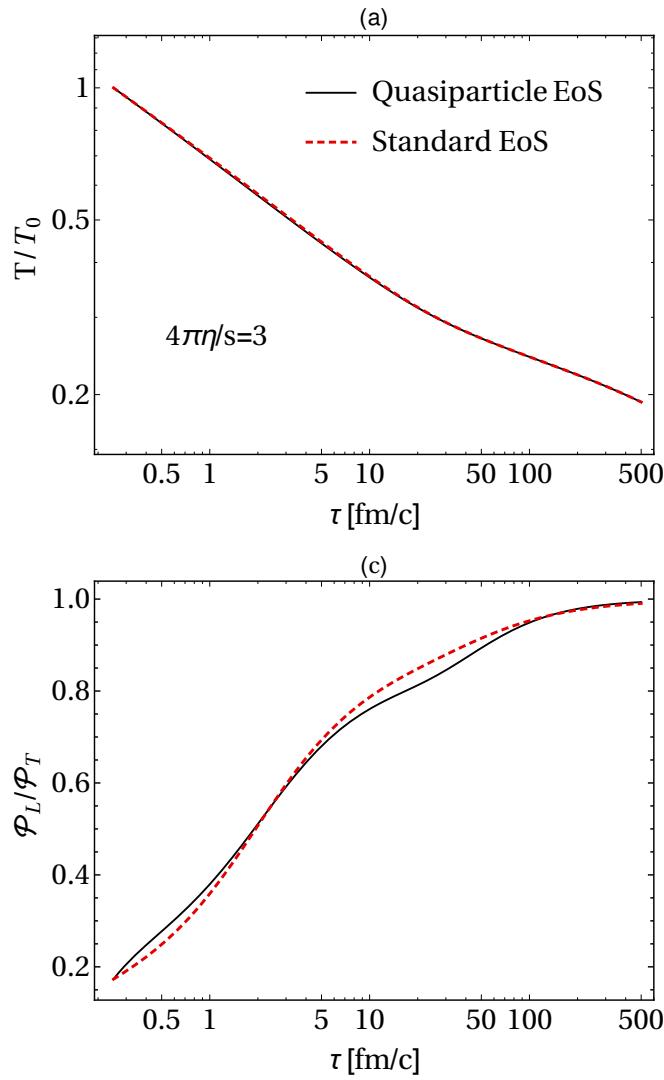
$$p^\mu \partial_\mu f + \frac{1}{2} \partial_i m^2 \partial_{(p)}^i f = -\mathcal{C}[f]$$

$$\partial_\mu B = -\frac{1}{2} \partial_\mu m^2 \int dP f(x, p)$$



Comparing the two methods

[Alqahtani, Nopoush, MS, 1509.02913]



Anisotropic Freezeout

[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, 1506.05278]
 [Alqahtani, Haque, Nopoush, Ryblewski, MS, forthcoming]

- Use ellipsoidal form for the dynamical equations and also for “anisotropic freezeout” at LO.
- From includes both shear and bulk corrections to the distribution function.
- Use energy density (scalar) to determine the freezeout hypersurface
 $\Sigma \rightarrow$ e.g. $T_{\text{FO}} = 150$ MeV

$$f(x, p) = f_{\text{iso}} \left(\frac{1}{\lambda} \sqrt{p_\mu \Xi^{\mu\nu} p_\nu} \right)$$

$$\Xi^{\mu\nu} = \underbrace{u^\mu u^\nu}_{\text{isotropic}} + \underbrace{\xi^{\mu\nu}}_{\text{anisotropy tensor}} - \underbrace{\Phi \Delta^{\mu\nu}}_{\text{bulk correction}}$$

$$\xi_{\text{LRF}}^{\mu\nu} \equiv \text{diag}(0, \xi_x, \xi_y, \xi_z)$$

$$\xi^\mu_\mu = 0 \quad u_\mu \xi^\mu_\nu = 0$$

$$\left(p^0 \frac{dN}{dp^3} \right)_i = \frac{\mathcal{N}_i}{(2\pi)^3} \int f_i(x, p) p^\mu d\Sigma_\mu ,$$

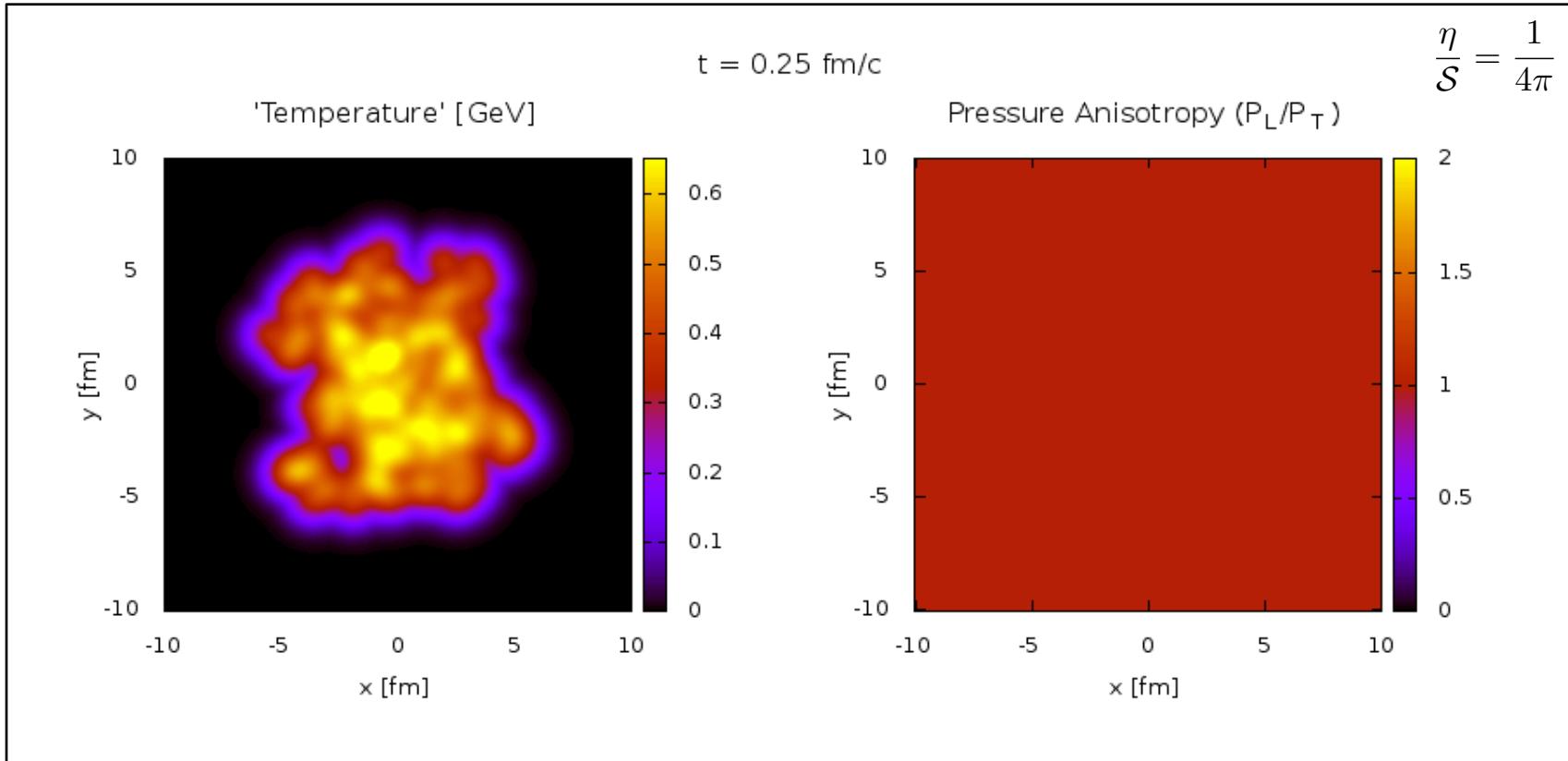
NOTE: Usual 2nd-order viscous hydro form

$$f(p, x) = f_{\text{eq}} \left[1 + (1 - af_{\text{eq}}) \frac{p_\mu p_\nu \Pi^{\mu\nu}}{2(\epsilon + P)T^2} \right]$$

$$f_{\text{eq}} = 1 / [\exp(p \cdot u/T) + a] \quad a = -1, +1, \text{ or } 0$$

This form suffers from the problem that the distribution function can be negative in some regions of phase space \rightarrow unphysical but unclear how important this is in the end

3+1d aHydro code



- Pb-Pb, $b = 7 \text{ fm}$ collision with Monte-Carlo Glauber initial conditions
 $T_0 = 600 \text{ MeV}$ @ $\tau_0 = 0.25 \text{ fm}/c$
- Left panel shows temperature and right shows pressure anisotropy

Focus on central collisions first

[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, 1506.05278]

- For central collisions we can describe the system with only two anisotropy parameters + bulk
- This maps directly to shear tensor components, e.g. rr and $\phi\phi$, plus a bulk correction
- Ignore bulk correction and assume boost invariance (for now)

$$\xi_{\text{LRF}}^{\mu\nu} \equiv \text{diag}(0, \xi_x, \xi_y, \xi_z)$$
$$\xi^\mu_{\mu} = 0 \quad u_\mu \xi^\mu_{\nu} = 0$$

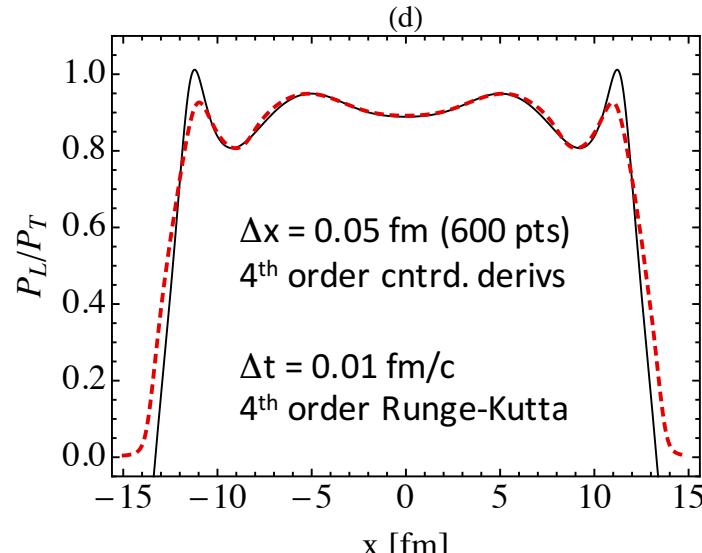
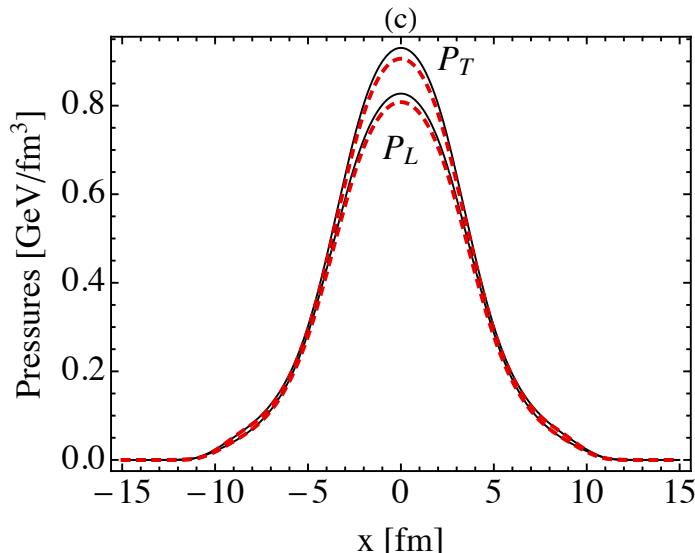
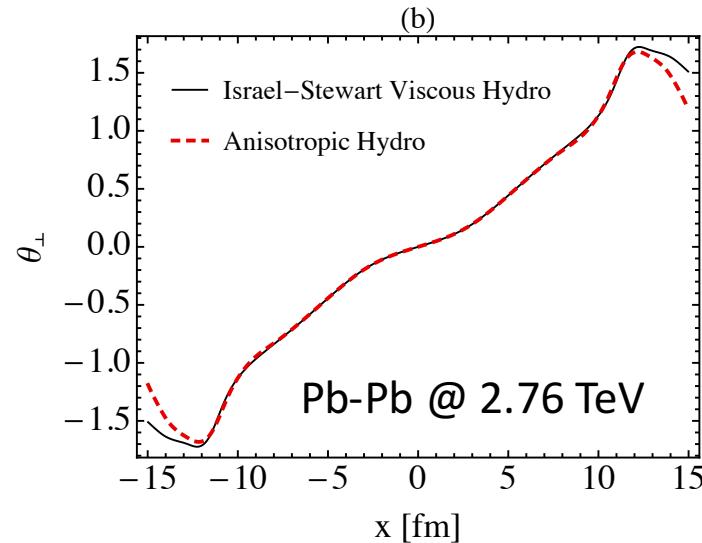
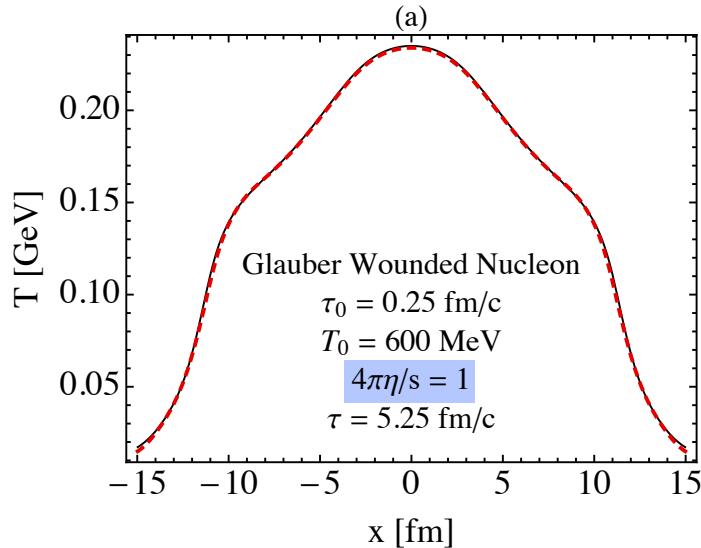
$$\Xi^{\mu\nu} = u^\mu u^\nu + \xi^{\mu\nu} - \Phi \cancel{\Delta}^{\mu\nu}$$

isotropic anisotropy bulk
tensor correction

$$D_u \varepsilon + \varepsilon \theta_u + P_x D_x \theta_\perp + P_y \frac{\sinh \theta_\perp}{r} + P_z \frac{\cosh \theta_\perp}{\tau} = 0,$$
$$D_x P_x + P_x \theta_x + \varepsilon D_u \theta_\perp - P_y \frac{\cosh \theta_\perp}{r} - P_z \frac{\sinh \theta_\perp}{\tau} = 0,$$
$$\frac{D_u \alpha_i}{\alpha_i} - \frac{1}{3} \sum_{j=x,y,z} \frac{D_u \alpha_j}{\alpha_j} - \sigma_i + \frac{1}{2\tau_{\text{eq}}} \left(1 - \frac{1}{\alpha_i^2}\right) \left(\frac{T}{\lambda}\right)^5 \frac{1}{\alpha_x \alpha_y \alpha_z} = 0, \quad i \in \{x, y\}.$$
$$\alpha_i = \sqrt{1 + \xi_i}$$

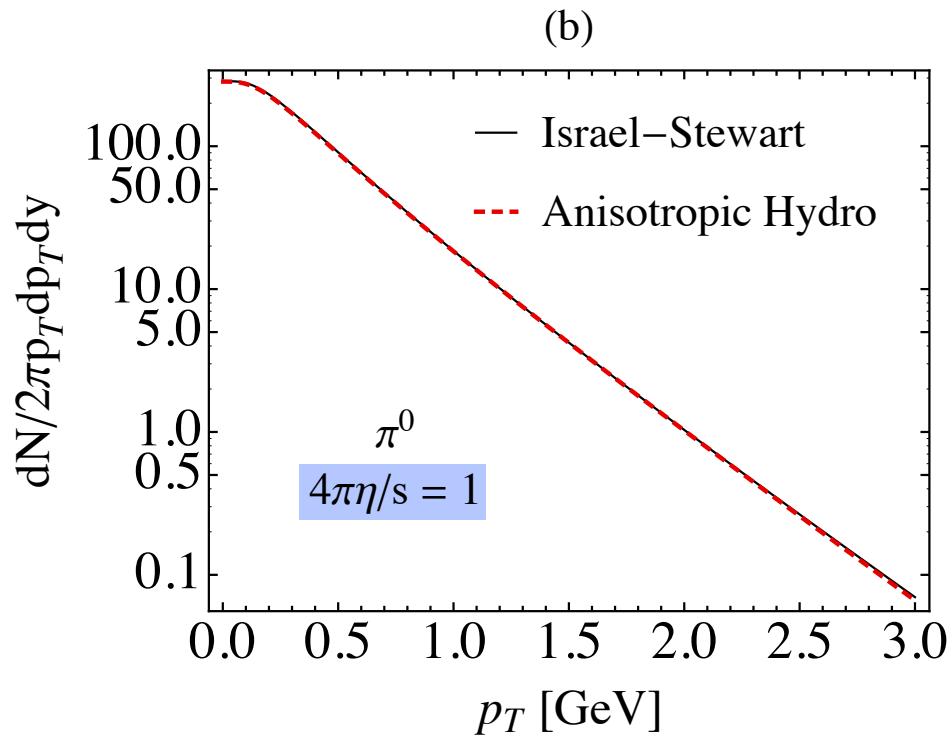
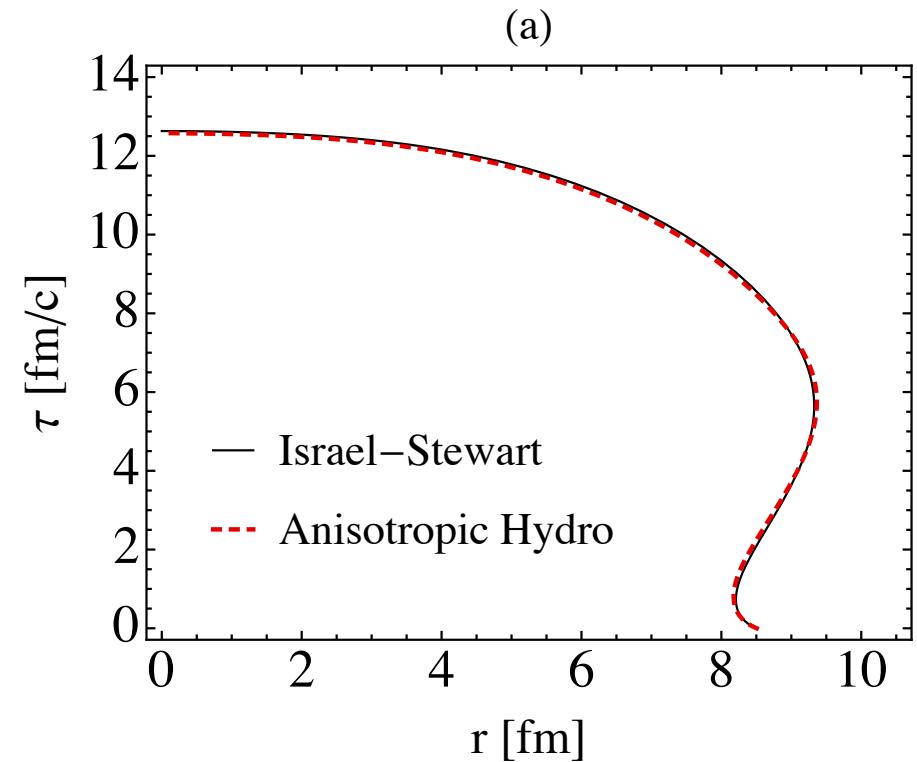
Results for central collisions (Pb-Pb)

[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, 1506.05278]



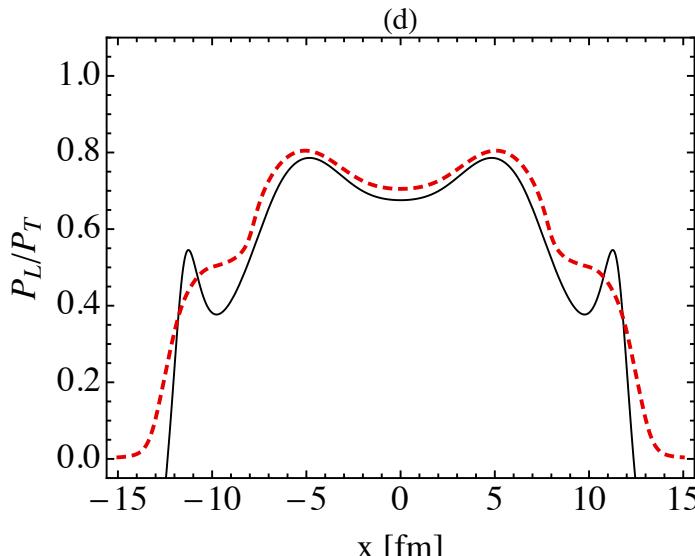
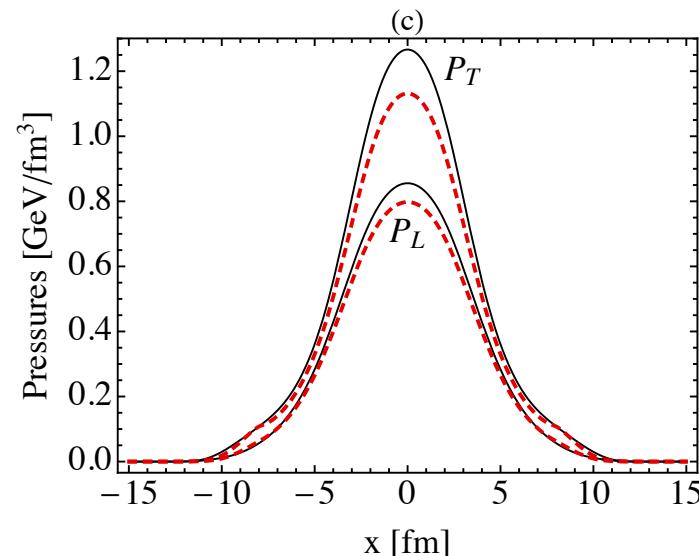
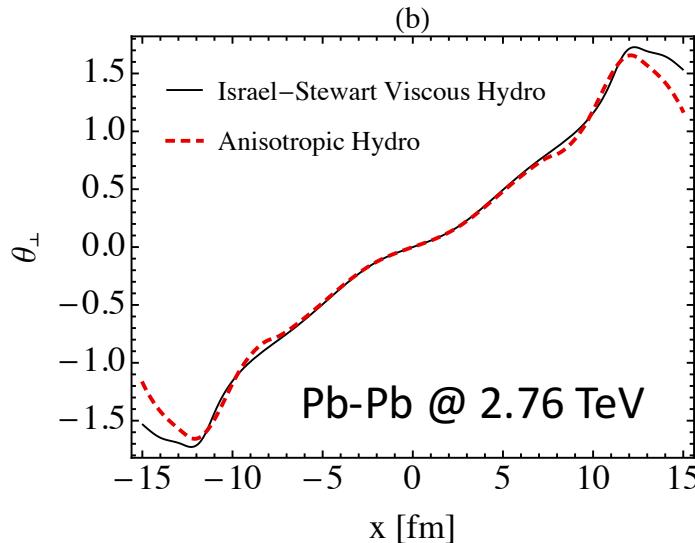
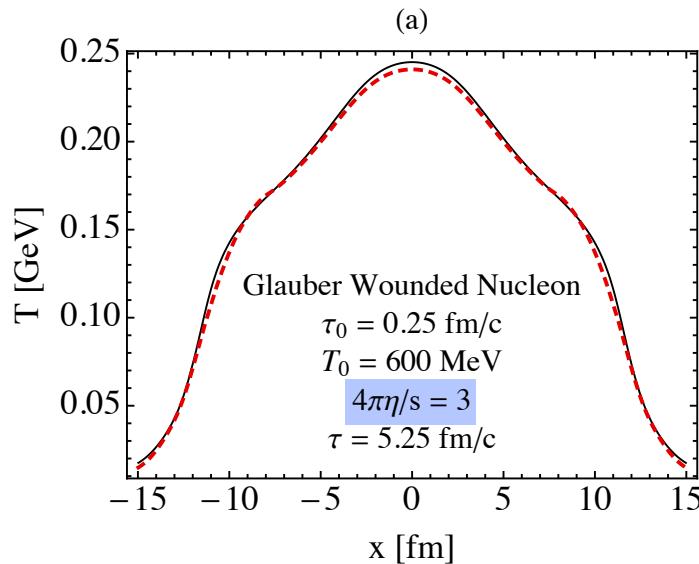
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[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, forthcoming]



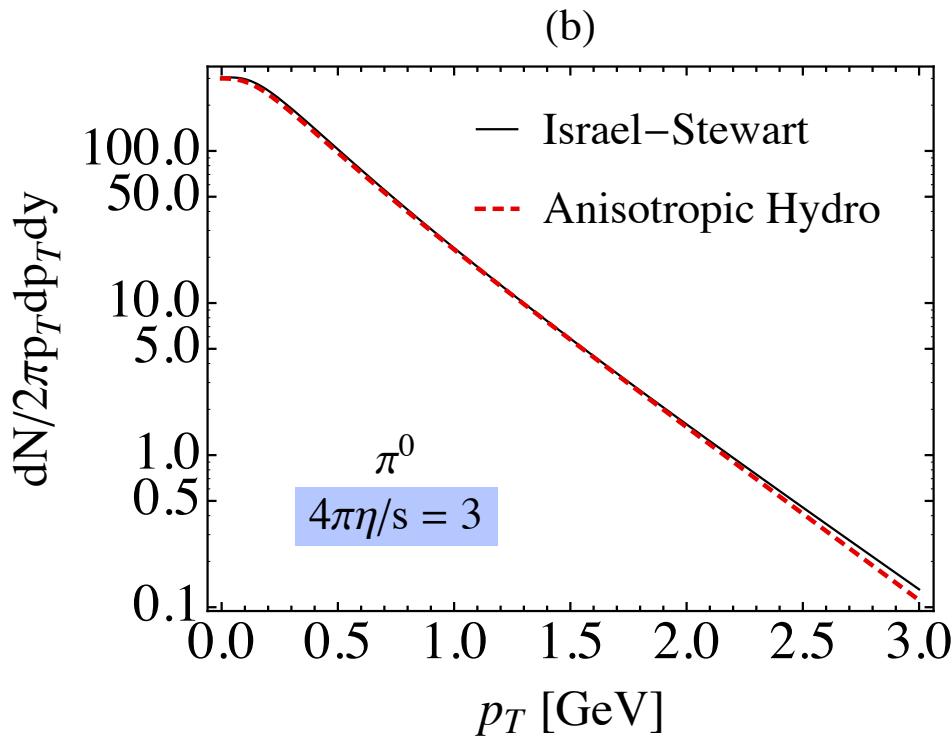
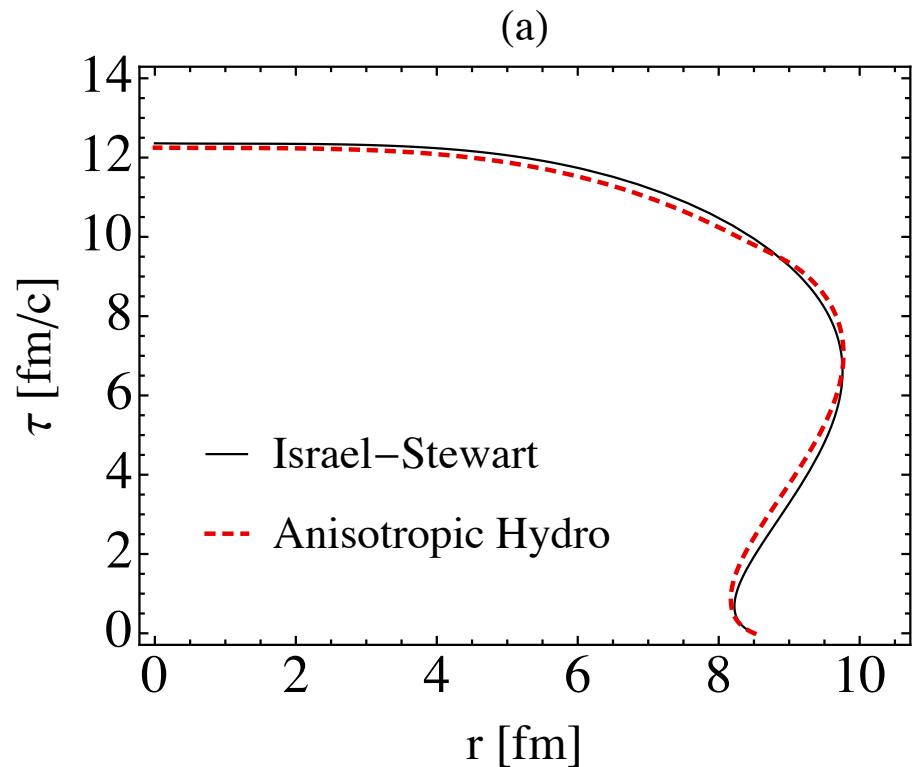
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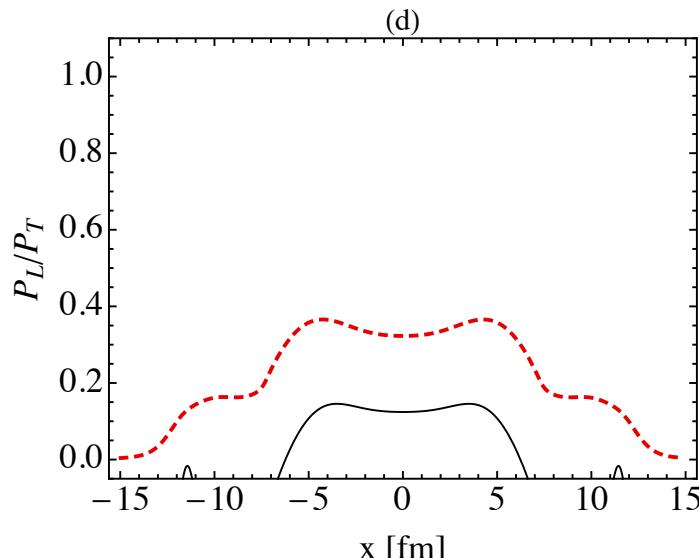
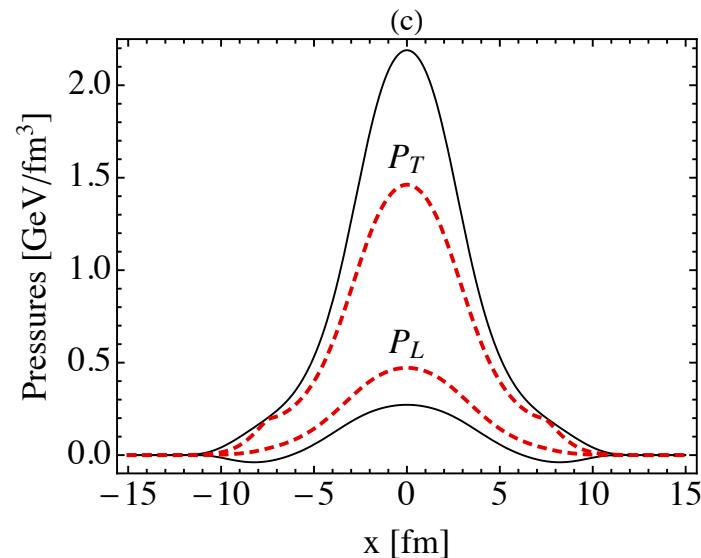
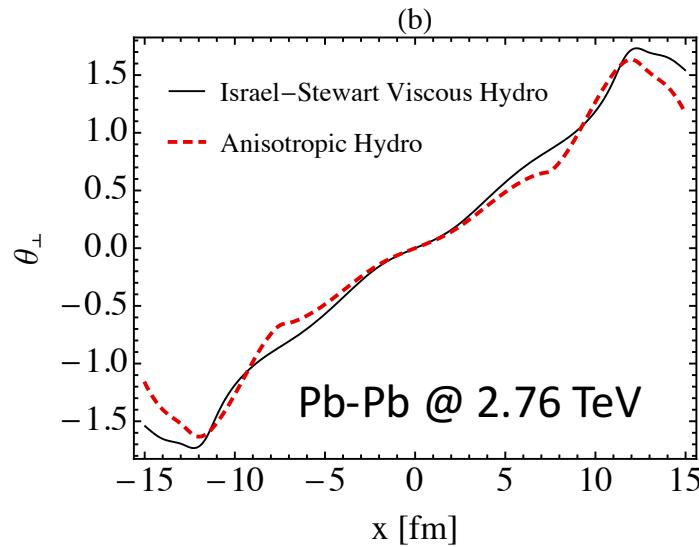
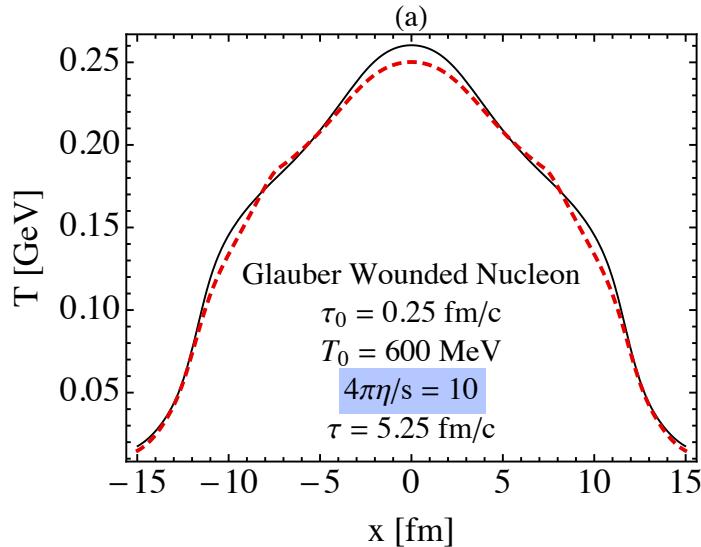
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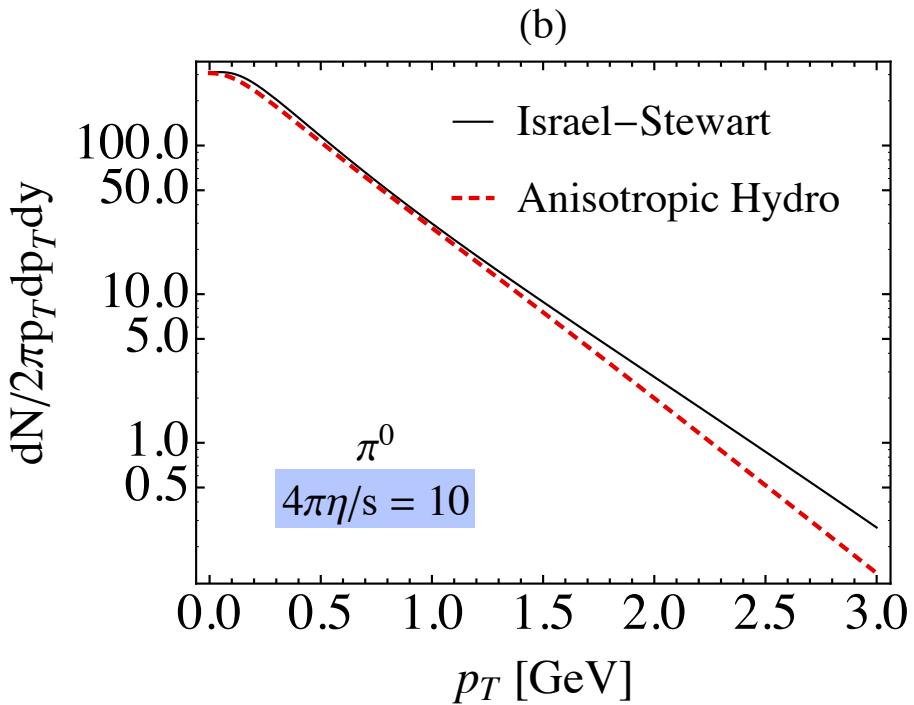
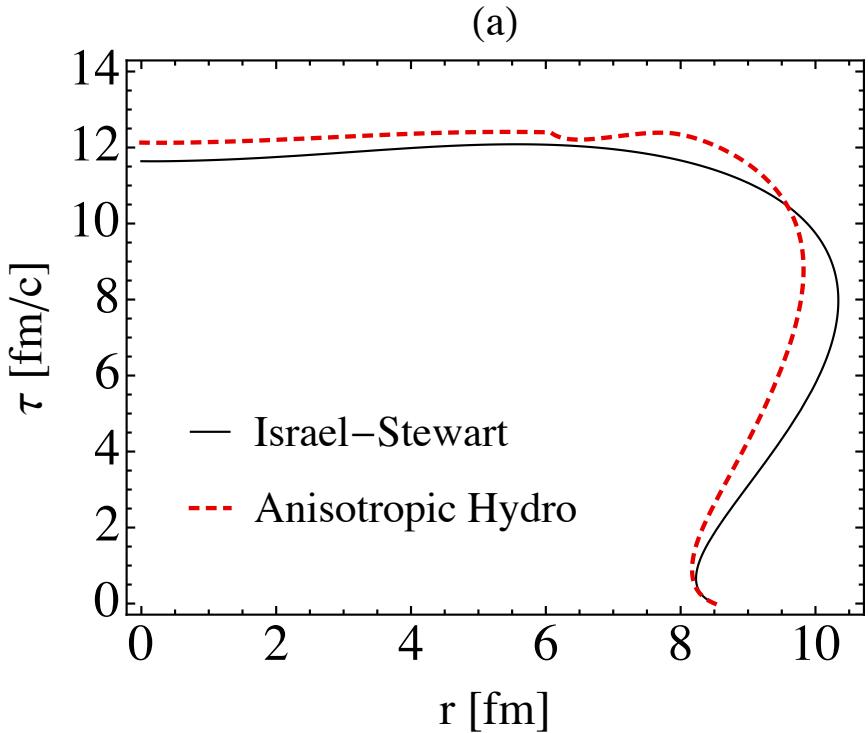
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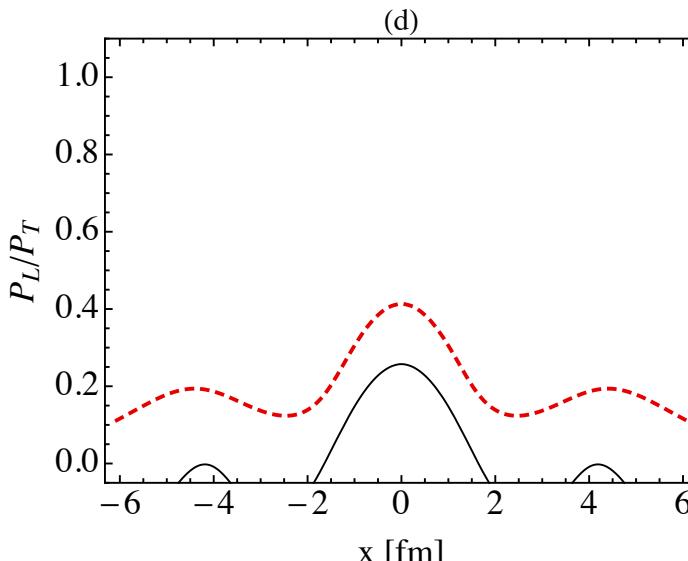
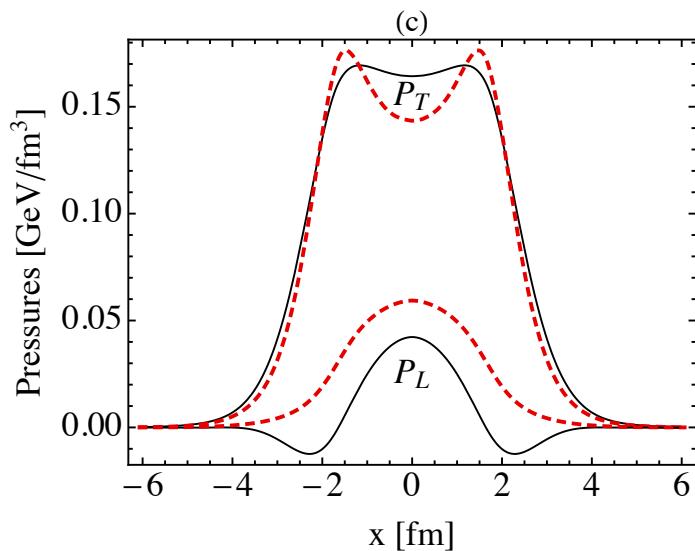
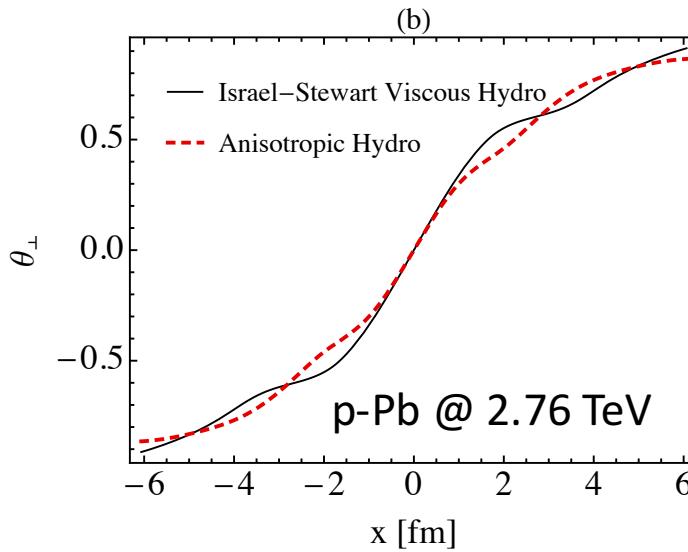
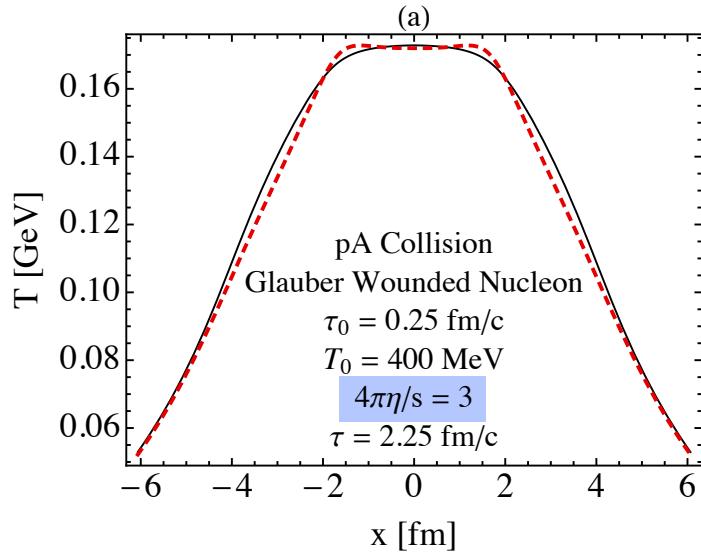
Results for central collisions (Pb-Pb)

[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, 1506.05278]



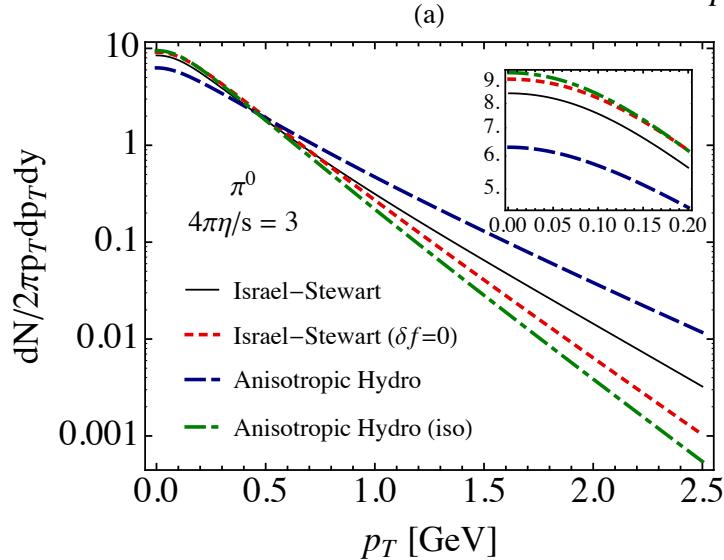
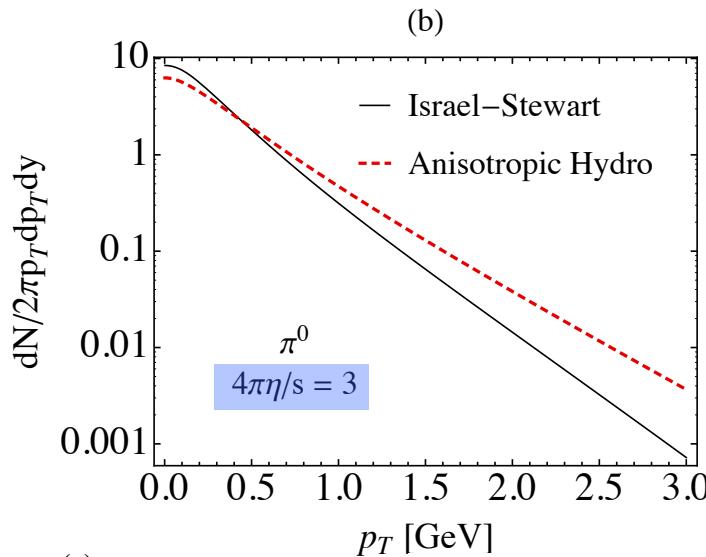
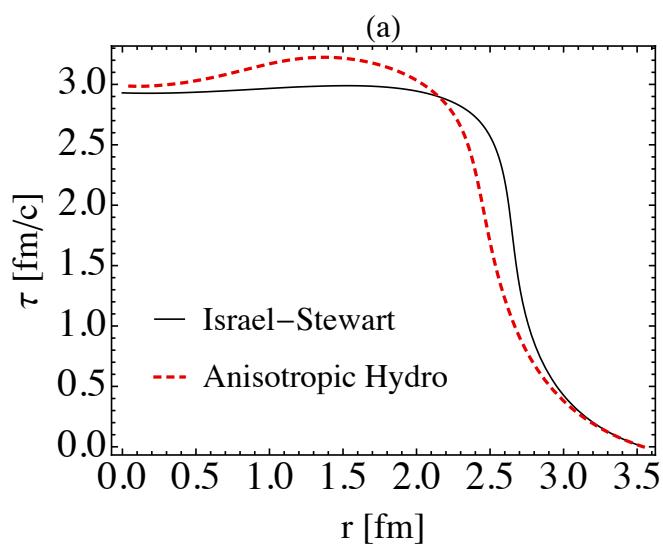
Results for central collisions (p-Pb)

[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, 1506.05278]



Results for central collisions (p-Pb)

[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, 1506.05278]



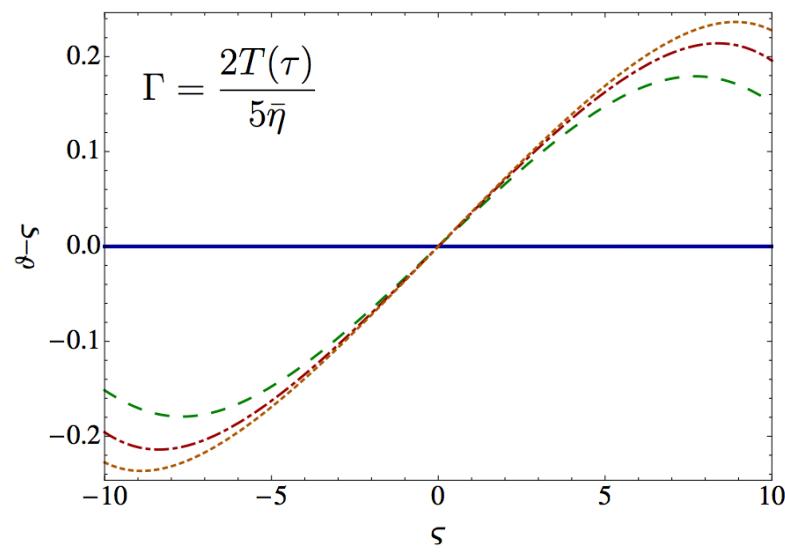
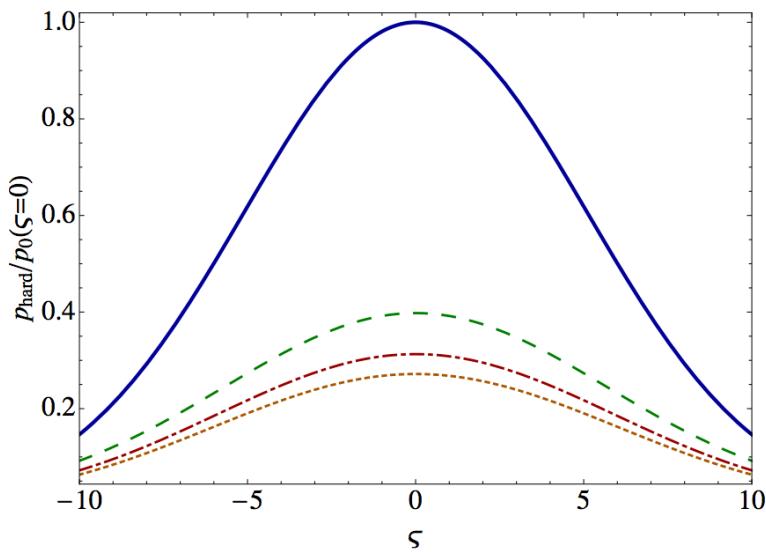
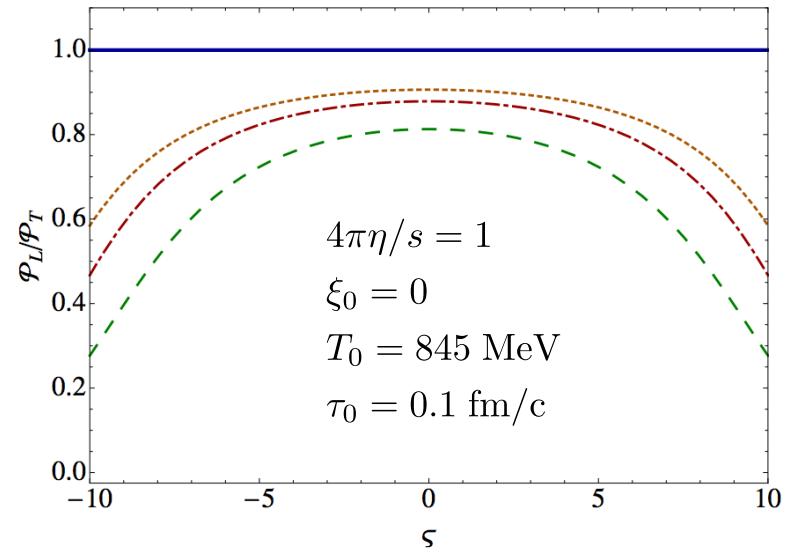
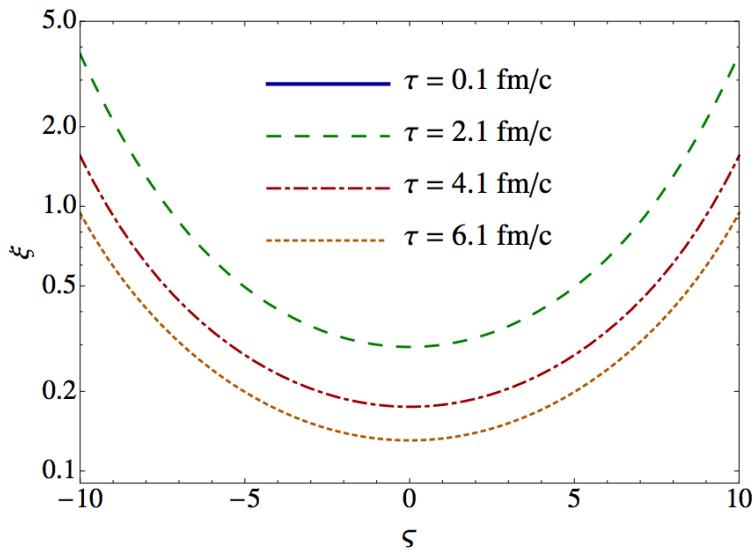
Conclusions and Outlook

- Anisotropic hydrodynamics builds upon prior advances in relativistic hydrodynamics in an attempt to create an even more quantitatively reliable tool.
- It incorporates some “facts of life” specific to the conditions generated in relativistic heavy ion collisions and, in doing so, optimizes the dissipative hydrodynamics approach.
- We now have a 3+1d LO aHydro code that is in testing.
- Today, I showed results of 1+1d LO aHydro and comparisons with 2nd-order viscous hydro.
- We find that, when they are supposed to agree (ideal and small viscosity limits), they do agree.
- For large shear viscosity or small system size, we see important corrections to the final spectra.

Backup Slides

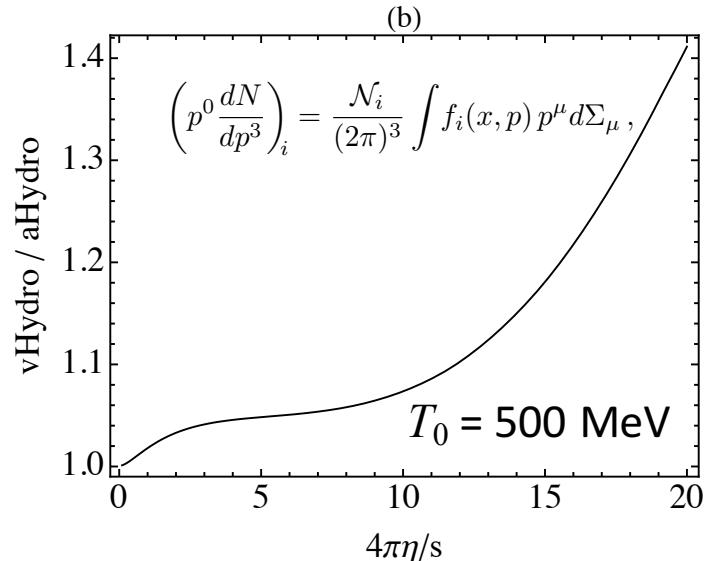
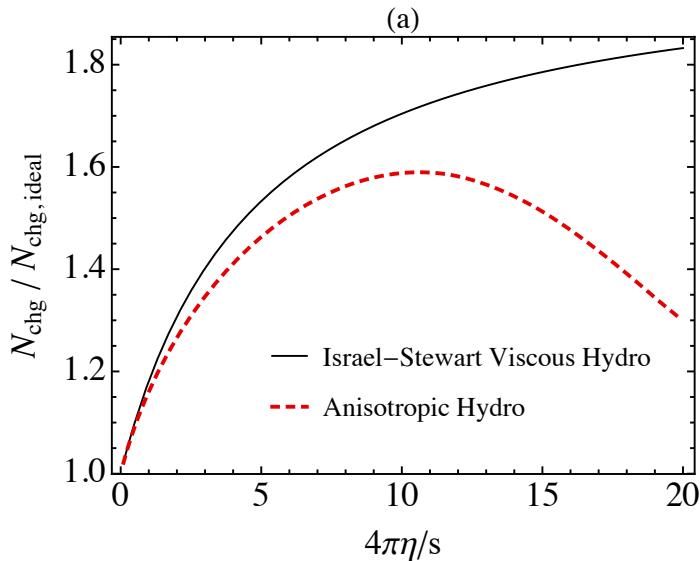
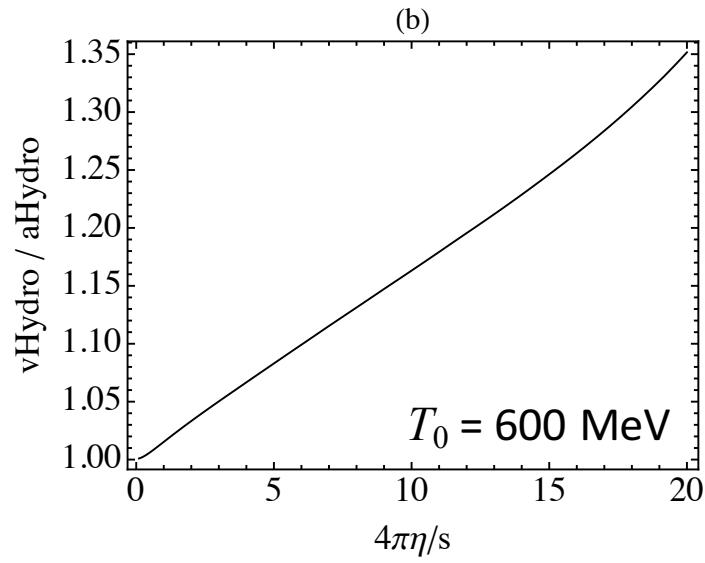
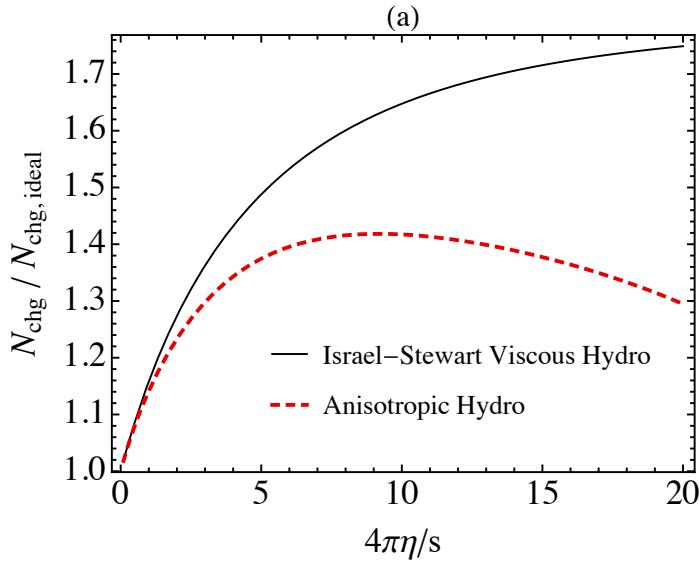
Rapidity dependence of pressure anisotropy

[Martinez and MS, 1011.3056]



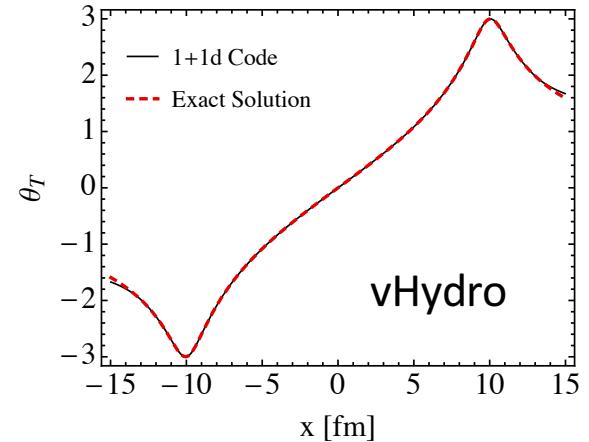
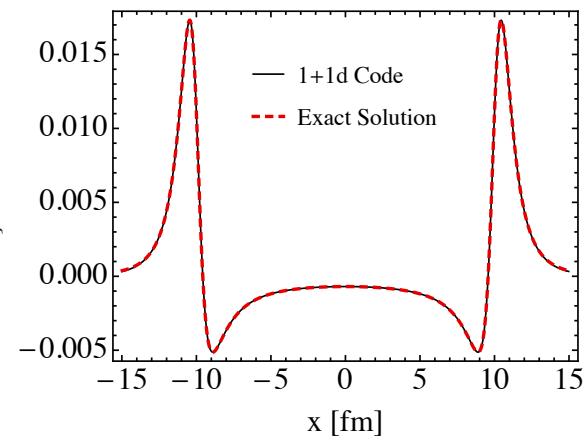
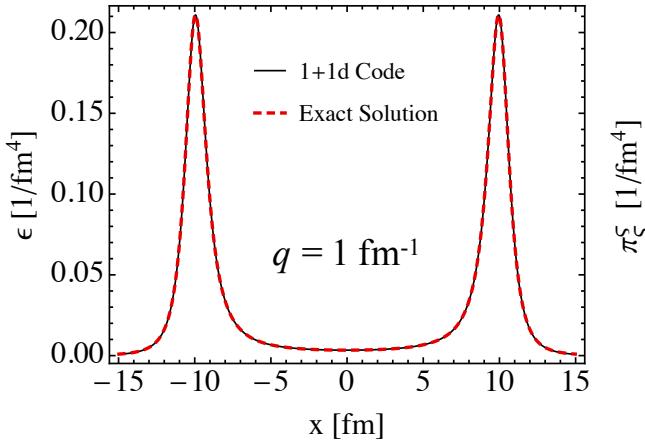
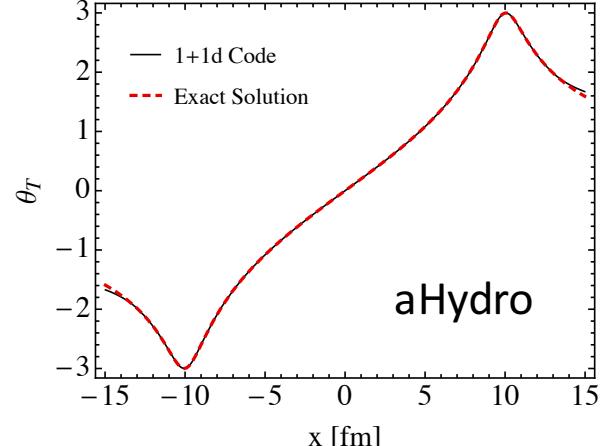
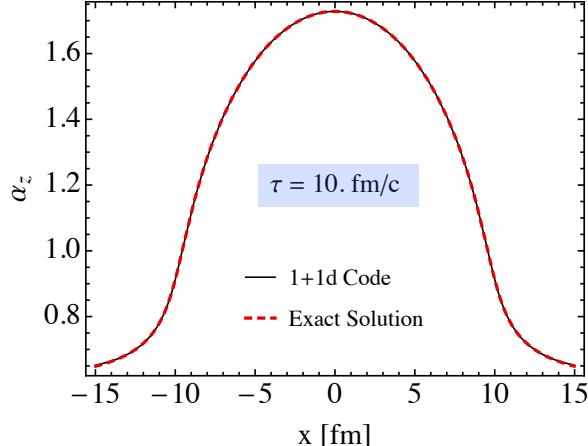
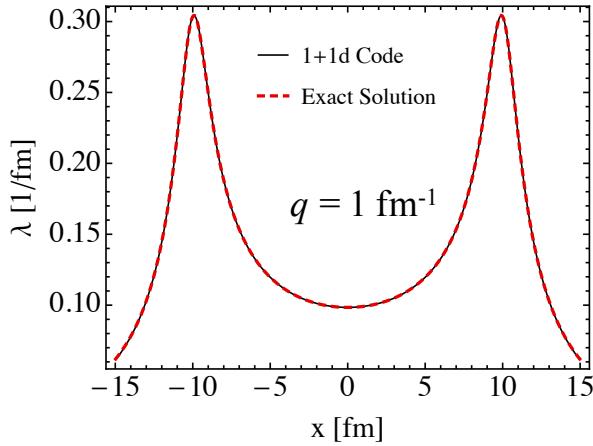
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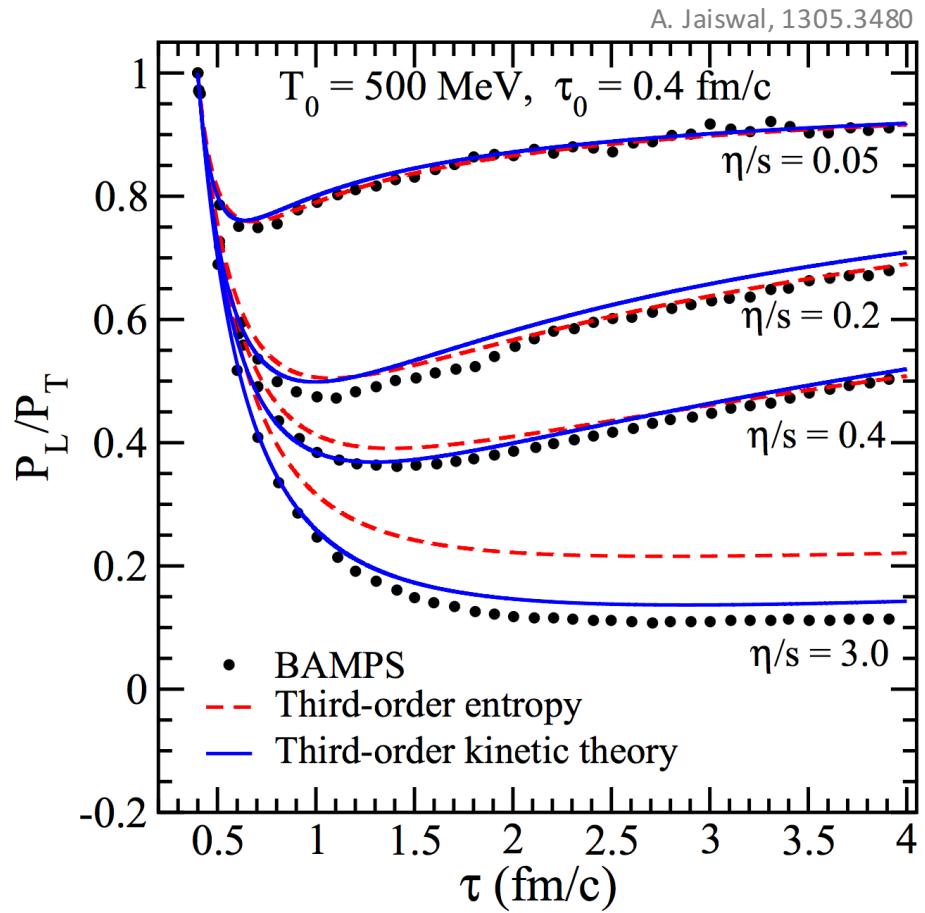
Gubser code tests – aHydro and vHydro

[Bazow, Heinz, Martinez, Nopoush, Ryblewski, MS, forthcoming]



Kinetic theory vs Hydro

- Kinetic theory can be pushed beyond its range of applicability and still has good agreement with hydrodynamic evolution!
- This is typical of a “good theory” in that, although it has some a priori limits, it can actually be applied further into the “forbidden zone” than one would naively guess.
- Right plot shows comparison of 3rd order viscous hydro results with a kinetic transport code with a tuned cross section.



BAMPS: A. El, Z. Xu and C. Greiner, Nucl. Phys. A 806, 287 (2008).

Azimuthally symmetric $T^{\mu\nu}$

$$T^{\mu\nu}(t, \mathbf{x}) = t_{00}g^{\mu\nu} + \sum_{i=1}^3 t_{ii}X_i^\mu X_i^\nu + \sum_{\substack{\alpha, \beta=0 \\ \alpha > \beta}}^3 t_{\alpha\beta}(X_\alpha^\mu X_\beta^\nu + X_\beta^\mu X_\alpha^\nu),$$

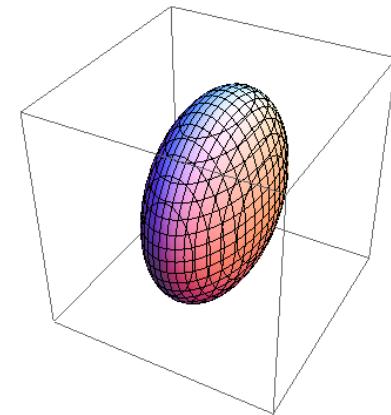
$$T_{\text{LRF}}^{00} = \mathcal{E} = t_{00},$$

$$T_{\text{LRF}}^{xx} = \mathcal{P}_\perp = -t_{00} + t_{11},$$

$$T_{\text{LRF}}^{yy} = \mathcal{P}_\perp = -t_{00} + t_{22},$$

$$T_{\text{LRF}}^{zz} = \mathcal{P}_L = -t_{00} + t_{33},$$

Assume, at leading order, rotational symmetry around p_z -axis in LRF



$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P}_\perp)u^\mu u^\nu - \mathcal{P}_\perp g^{\mu\nu} + (\mathcal{P}_L - \mathcal{P}_\perp)z^\mu z^\nu,$$

LO Spheroidal Distribution

- Consider a conformal system to start with
- In the conformal (massless) limit all bulk observables factorize into a product of two functions

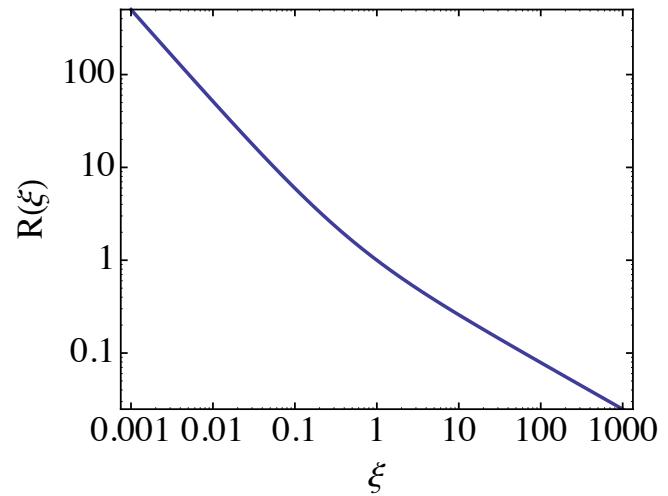
$$n(\Lambda, \xi) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} f_{\text{aniso}} = \frac{n_{\text{iso}}(\Lambda)}{\sqrt{1 + \xi}}$$

$$\mathcal{E}(\Lambda, \xi) = T^{\tau\tau} = \mathcal{R}(\xi) \mathcal{E}_{\text{iso}}(\Lambda)$$

$$\mathcal{P}_\perp(\Lambda, \xi) = \frac{1}{2} (T^{xx} + T^{yy}) = \mathcal{R}_\perp(\xi) \mathcal{P}_{\text{iso}}(\Lambda)$$

$$\mathcal{P}_L(\Lambda, \xi) = -T_\zeta^\zeta = \mathcal{R}_L(\xi) \mathcal{P}_{\text{iso}}(\Lambda)$$

$$\begin{aligned}\mathcal{R}(\xi) &\equiv \frac{1}{2} \left(\frac{1}{1 + \xi} + \frac{\arctan \sqrt{\xi}}{\sqrt{\xi}} \right) \\ \mathcal{R}_\perp(\xi) &\equiv \frac{3}{2\xi} \left(\frac{1 + (\xi^2 - 1)\mathcal{R}(\xi)}{\xi + 1} \right) \\ \mathcal{R}_L(\xi) &\equiv \frac{3}{\xi} \left(\frac{(\xi + 1)\mathcal{R}(\xi) - 1}{\xi + 1} \right)\end{aligned}$$



0+1d case – new Bjorken equations

0th Moment of Boltzmann EQ

$$\partial_\alpha N^\alpha \neq 0$$

$$\frac{1}{1+\xi} \partial_\tau \xi - \frac{2}{\tau} - 6 \partial_\tau \log \Lambda = 2\Gamma \left[1 - \mathcal{R}^{3/4}(\xi) \sqrt{1+\xi} \right]$$

1st Moment of Boltzmann EQ

$$\partial_\alpha T^{\alpha\beta} = 0$$

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)} \partial_\tau \xi + 4 \partial_\tau \log \Lambda = \frac{1}{\tau} \left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1 \right]$$

In relaxation time approximation

$$\Gamma = \frac{2T(\tau)}{5\bar{\eta}} = \frac{2\mathcal{R}^{1/4}(\xi)\Lambda}{5\bar{\eta}}$$

[M. Martinez and MS, 1007.0889]

$$\mathcal{R}(\xi) \equiv \frac{1}{2} \left(\frac{1}{1+\xi} + \frac{\arctan \sqrt{\xi}}{\sqrt{\xi}} \right)$$
$$\mathcal{E}(\Lambda, \xi) = \mathcal{R}(\xi) \mathcal{E}_{\text{iso}}(\Lambda)$$

Linearized Equations

If we expand the energy-momentum tensor to linear order in the anisotropy parameter and match to 2nd-order viscous hydro, we find

$$\frac{\Pi}{\mathcal{E}_{\text{eq}}} = \frac{8}{45} \xi + \mathcal{O}(\xi^2)$$

If we similarly expand the coupled nonlinear differential equations to lowest order in the anisotropy parameter, we obtain

$$\begin{aligned}\partial_\tau \mathcal{E} &= -\frac{\mathcal{E} + \mathcal{P}}{\tau} + \frac{\Pi}{\tau} \\ \partial_\tau \Pi &= -\frac{\Pi}{\tau_\pi} + \frac{4}{3} \frac{\eta}{\tau_\pi \tau} - \frac{4}{3} \frac{\Pi}{\tau}\end{aligned}$$

$$\begin{aligned}\Gamma &= \frac{2}{\tau_\pi} \\ \tau_\pi &= \frac{5}{4} \frac{\eta}{\mathcal{P}}\end{aligned}$$

- Reproduces 2nd-order viscous hydro in the small anisotropy limit
- For the 3+1d proof see the recent paper of L. Tinti [1411.7268]
- Also reproduces the free streaming limit!

Including Transverse Dynamics

W. Florkowski and R. Ryblewski, 1103.1260
M. Martinez, R. Ryblewski, and MS, 1204.1473

- Allowing variables to depend on x and y while still assuming boost-invariance, we obtain the “2+1d” dimensional AHYDRO equations
- Conformal system \rightarrow four equations for four variables u_x , u_y , ξ , and Λ .

0th moment

$$Dn + n\theta = J_0 .$$

$$D \equiv u^\mu \partial_\mu ,$$

$$\theta \equiv \partial_\mu u^\mu ,$$

$$u_0 = \sqrt{1 + u_x^2 + u_y^2}$$

1st moment

$$D\mathcal{E} + (\mathcal{E} + \mathcal{P}_\perp)\theta + (\mathcal{P}_L - \mathcal{P}_\perp)\frac{u_0}{\tau} = 0 ,$$

$$(\mathcal{E} + \mathcal{P}_\perp)Du_x + \partial_x \mathcal{P}_\perp + u_x D\mathcal{P}_\perp + (\mathcal{P}_\perp - \mathcal{P}_L)\frac{u_0 u_x}{\tau} = 0 ,$$

$$(\mathcal{E} + \mathcal{P}_\perp)Du_y + \partial_y \mathcal{P}_\perp + u_y D\mathcal{P}_\perp + (\mathcal{P}_\perp - \mathcal{P}_L)\frac{u_0 u_y}{\tau} = 0 .$$

Estimating Anisotropy – Viscous hydro

- To get a feeling for the magnitude of pressure anisotropies to expect, let's consider the Navier-Stokes limit for a conformal gas

$$\left(\frac{P_L}{P_T}\right)_{\text{NS}} = \frac{P_{\text{eq}} + \pi_{\text{NS}}^{zz}}{P_{\text{eq}} + \pi_{\text{NS}}^{xx}} = \frac{3\tau T - 16\bar{\eta}}{3\tau T + 8\bar{\eta}}$$

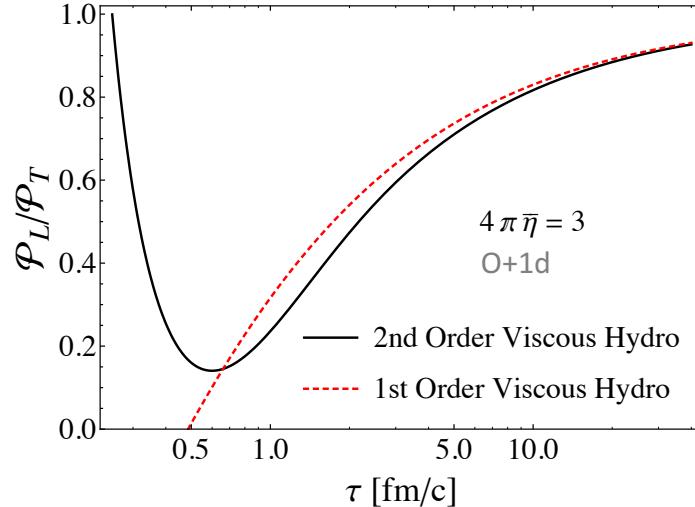
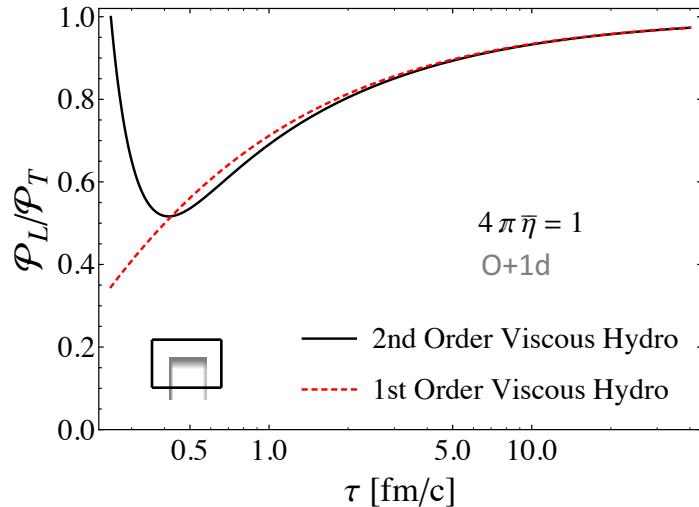
$$\bar{\eta} = \frac{\eta}{S}$$

$$\pi_{\text{NS}}^{zz} = -2\pi_{\text{NS}}^{xx} = -2\pi_{\text{NS}}^{yy} = -4\eta/3\tau$$

- P_L/P_T decreases with increasing η/S or decreasing T
- Assume $\eta/S = 1/4\pi$ in order to get an upper bound on the anisotropy
- Using RHIC initial conditions one obtains $P_L/P_T \leq 0.5$
- Using LHC initial conditions one obtains $P_L/P_T \leq 0.35$
- Negative P_L at large η/S or low temperatures!?
- Even worse, the one-particle distribution function can become negative in some regions of phase space!

Estimating Early-time Pressure Anisotropy

- CGC @ leading order predicts negative → approximately zero longitudinal pressure
- QGP scattering + plasma instabilities work to drive the system towards isotropy on the fm/c timescale, but don't seem to fully restore it
- Viscous hydrodynamics predicts early-time anisotropies $\leq 0.35 \rightarrow 0.5$
- AdS-CFT dynamical calculations in the strong coupling limit predict anisotropies of ≤ 0.3



Estimating Anisotropy – AdS/CFT

- In 0+1d case there are numerical solutions of Einstein's equations to compare with.
[Heller, Janik, and Witaszczyk, 1103.3452]
- They studied a wide variety of initial conditions and found a kind of universal lower bound for the “hydronization” time

RHIC 200 GeV/nucleon:
 $T_0 = 350 \text{ MeV}$, $\tau_0 > 0.35 \text{ fm/c}$

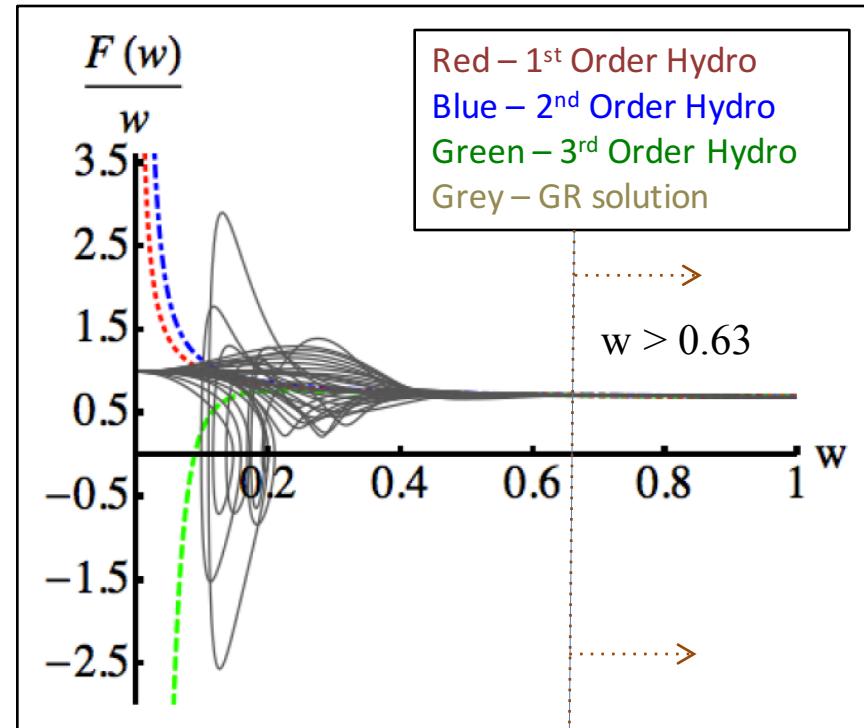
LHC 2.76 TeV/nucleon:
 $T_0 = 600 \text{ MeV}$, $\tau_0 > 0.2 \text{ fm/c}$

$$\langle T_{\tau\tau} \rangle \equiv \varepsilon(\tau) \equiv N_c^2 \cdot \frac{3}{8}\pi^2 \cdot T_{eff}^4 .$$

$$w = T_{eff} \cdot \tau$$

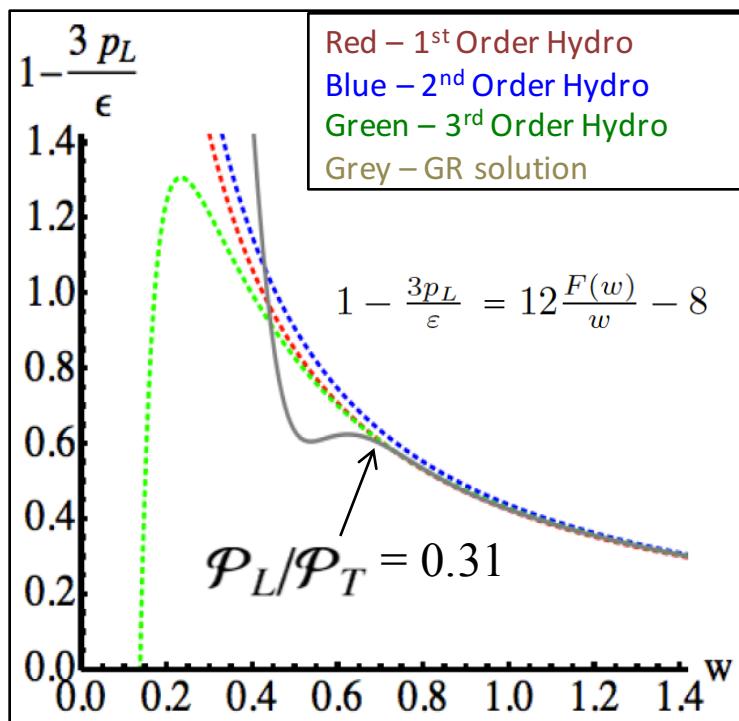
$$\frac{\tau}{w} \frac{d}{d\tau} w = \frac{F_{hydro}(w)}{w},$$

F_{hydro} known up to 3rd order hydro analytically



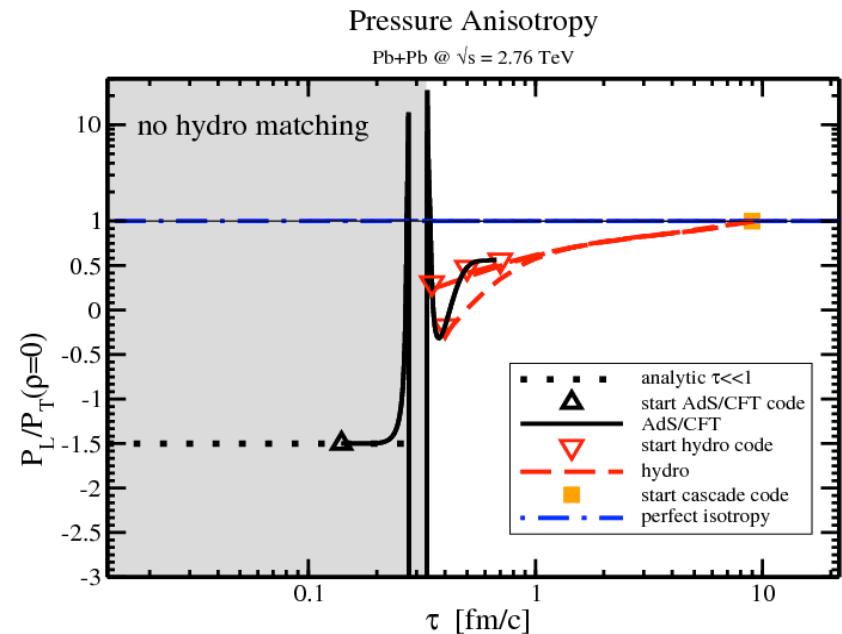
N=4 SUSY using AdS/CFT

However, at that time the system is not isotropic and remains anisotropic for the entirety of the evolution



Another AdS/CFT numerical GR paper which includes transverse expansion reaches a similar conclusion

[van der Schee et al. 1307.2539]



See also J. Casalderrey-Solana et al.
arXiv:1305.4919